TOWARDS THE COHOMOLOGY OF AUGMENTATION VARIETIES OF LEGENDRIAN TANGLES

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Abstract. Associated to any Legendrian tangle, the augmentation variety (with fixed boundary conditions), hence its mixed Hodge structure on the compactly supported cohomology, is a Legendrian isotopy invariant up to a normalization. Induced from the ruling decomposition of the variety, there's a spectral sequence converging to the MHS. As an application, we show that the variety is of Hodge-Tate type, and show a vanishing result on the cohomology. We also do some example computations of MHSs. In the end, we conjecture that the ruling decomposition for the full augmentation variety of acyclic augmentations is a Whitney stratification, and the geometric partial order via inclusions of stratum closures admits an explicit combinatorial description. We verify the conjecture for the cases of trivial and elementary Legendrian tangles.

Contents

Introduction

A powerful modern Legendrian isotopy invariant in the study of Legendrian knots Λ in the standard contact three space $\mathbb{R}^3 = J^1\mathbb{R}_x$, is the Chekanov-Eliashberg differential graded algebra (C-E DGA) $\mathcal{A}(\Lambda) = \mathcal{A}(\mathbb{R}^3, \Lambda)$. The C-E DGAs are special cases of the more general Legendrian contact homology differential graded algebras (LCH DGA) $\mathcal{A}(V, \Lambda)$, associated to a Legendrian submanifold Λ in a contact manifold V. The algebra $\mathcal{A}(V, \Lambda)$ is generated by the Reeb chords of Λ, whose differential counts certain holomorphic disks in the symplectization $\mathbb{R} \times V$, with boundary on the Lagrangian cylinder $\mathbb{R} \times V$ and meeting the Reeb chords at some punctures [Eli98,EGH00]. The LCH DGAs $\mathcal{A}(V, \Lambda)$, up to homotopy equivalence, are Legendrian isotopy invariants.

In the case of Legendrian knots Λ, the DGA $\mathcal{A}(\Lambda)$ also admits a combinatorial description [Che02,ENS02,Ng03]. More recently, the construction is extended to obtain LCH DGAs $\mathcal{A}(T)$ for any Legendrian tangles T in the 1-jet bundle $J^1U \hookrightarrow J^1\mathbb{R}_x$, with $U \hookrightarrow \mathbb{R}$ an open interval [Siv11,NRS$^+$15,Su17]. The LCH DGAs $\mathcal{A}(T|_V)$ satisfy a co-sheaf/van-Kampen property over open $V \hookrightarrow U$, hence behave like 'fundamental groups'. The invariance of the DGAs $\mathcal{A}(T)$ up to homotopy equivalence ensures we obtain Legendrian isotopy invariants by studying the Hodge theory of their 'representation varieties' (called augmentation varieties). In particular, the study of the augmentation varieties is like that of character varieties, for example, as in [HRV08]. In the case of Legendrian tangles T, the natural objects to consider are augmentation varieties with fixed boundary conditions $\mathrm{Aug}_m(T, \rho_L, \rho_R; k)$ [Su17]. In particular, their point-counting over finite fields, or equivalently by [HRV08, Katz's appendix], weight polynomials, recover the ruling polynomials $< \rho_L|R_T^m(z)|\rho_R >$. The latter are invariants defined combinatorially via the decomposition of the front diagrams of T, and satisfy a composition axiom, reflecting the sheaf property of augmentation varieties induced from the co-sheaf property of the LCH DGAs $\mathcal{A}(T)$. Moreover, the sheaf property allows one to derive a decomposition (the ruling/Henry-Rutherford decomposition) of the augmentation varieties [Su17] (see also [HR15] for the case of Legendrian knots): $\mathrm{Aug}_m(T, \rho_L, \rho_R; k) = \sqcup_{\rho \in \mathrm{NR}_T^m(\rho_L, \rho_R)} \mathrm{Aug}_m^{\rho}(T; k)$, where each piece $\mathrm{Aug}_m^{\rho}(T; k)$ is of the simple form $(k^*)^{a(\rho)} \times k^{b(\rho)}$.

Organization and results. In this article, we pursue a study of the mixed Hodge structure on the (compactly supported) cohomology of the augmentation varieties $\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$. The organization and results of this article are as follows: In Section 1, we review some necessary background on Legendrian knot theory; In Section 3, via a tangle approach, we establish the 'invariance' of augmentation varieties with fixed boundary conditions $X = \mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$ (Theorem 3.10), in particular, their mixed Hodge structures (Corollary 3.11). In Section 2.1, we use the ruling decomposition (Theorem 1.29) in [Su17] to derive a spectral sequence converging to the mixed Hodge structure on X (Lemma 2.4). In Section 2.2, we use the spectral sequence to show that the augmentation variety X is of Hodge-Tate type (Proposition 2.8), and $H_c^*(X) = 0$ if $* < C$ where $C = a(\rho) + 2b(\rho)$ is a constant depending only T and the boundary conditions (ρ_L, ρ_R) (Proposition 2.9). We also point out the 'invariance' of the mixed Hodge structure associated to 1st page of the spectral sequence by forgetting the differential (Lemma 2.10). In Section 2.3, we compute some examples of mixed Hodge structures of the augmentation varieties $\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$. Finally, in Section 4, we study the combinatorics of the ruling decomposition associated to the augmentation varieties $\mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$. We conjecture that, the ruling decomposition for the full augmentation variety $\mathrm{Aug}_m^a(T;k)$ of acyclic augmentations is a Whitney stratification, and its geometric partial order via inclusions of stratum closures admits an explicit combinatorial description (Conjecture 4.17). We verify the conjecture in the cases of the 'building blocks' of Legendrian tangles: the trivial and elementary Legendrian tangles in Section 4.1 (Corollary 4.6) and Section 4.2 (Lemma 4.15).

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1. Background

1.1. **Legendrian tangles.**

1.1.1. *Basic definitions.* Let $U = (x_L, x_R)$ be an open interval in $\mathbb{R}_x$ for $-\infty \leq x_L < x_R \leq \infty$, consider the standard contact 3-manifold $J^1U = T^*U \times \mathbb{R}_z \subset J^1\mathbb{R}_x = \mathbb{R}^3_{x,y,z}$, with contact form $\alpha = dz - ydx$. The Reeb vector field of α is then $R_\alpha = \partial_z$. As in [Su17], we consider (one-dimensional) Legendrian submanifolds (termed as *Legendrian tangles*) T in J^1U, which are closed in J^1U and transverse to the boundary $\partial J^1\overline{U}$. In the special case when $x_L = -\infty, x_R = \infty$, then $U = \mathbb{R}_x$ and Legendrian tangles are *Legendrian knots/links* in $\mathbb{R}^3_{x,y,z}$ in the usual sense. The *front* and *Lagrangian projections* of T are $\pi_{xz}(T)$ and $\pi_{xy}(T)$ respectively, with the obvious projections $\pi_{xz} : J^1U \to U \times \mathbb{R}_z$ and $\pi_{xy} : J^1U \to T^*U = U \times \mathbb{R}_y$.

We say 2 Legendrian tangles in J^1U are *Legendrian isotopic* if there's an isotopy between them along Legendrian tangles in J^1U. Note that during the Legendrian isotopy, we require the *ordering* via z-coordinates of the left (resp. right) endpoints is preserved. That is, for two (say, left) end-points p_1, p_2, they necessarily have the common x-coordinate x_L, take any path γ in $\partial J^1(\overline{U})$ from p_2 to p_1, then we say $p_1 > p_2$ if $z(p_1) - z(p_2) = \int_\gamma \alpha > 0$.

1.1.2. *Front diagrams.* We will always assume the Legendrian tangle $T \subset J^1U$ is in a generic position inside its Legendrian isotopy class. So, the front projection $\pi_{xz}(T)$ gives a *(tangle) front diagram*, i.e. an immersion of a finite union of circles and intervals into $U \times \mathbb{R}_z$ away from finitely many points (cusps), which is also an embedding away from finitely many points (cusps and transversal crossings), such that it has no vertical tangents, sends the boundaries of the intervals to the boundary $\partial\overline{U}\times\mathbb{R}_z$ and is transverse to the boundary. The significance of front diagrams is that, any Legendrian tangle is uniquely determined by its front projection[1], with the y-coordinate recovered from the x and z-coordinate, via the Legendrian condition $dz - ydx = 0 \Rightarrow y = dz/dx$. Note also that, near each crossing of a front diagram, the strand of the lesser slope is always the over-strand.

[1]From now on, we will make no distinction between Legendrian tangles and their front diagrams.

Given a front diagram $\pi_{xz}(T)$ in J^1U , the *strands* of $\pi_{xz}(T)$ are the maximally immersed connected submanifolds, the *arcs* of $\pi_{xz}(T)$ are the maximally embedded connected submanifolds and the *regions* are the maximal connected components of the complement of $\pi_{xz}(T)$ in $U\times\mathbb{R}_z$.

We say a front diagram in $U\times\mathbb{R}_z$ is *plat* if the crossings have distinct x-coordinates, all the left cusps have the same x-coordinate, wihch is different from those of the crossings and right cusps, and likewise for the right cusps. We say a front diagram is *nearly plat*, if it's a perturbation of a plat front diagram, so that the crossings and cusps all have distinct x-coordinates. We can always make the front diagram $\pi_{xz}(T)$ (nearly) plat by smooth isotopies and Legendrian Reidemeister II moves (see FIGURE 1.2).

1.1.3. *Resolution construction.* Given any front diagram $\pi_{xz}(T)$ in $U\times\mathbb{R}_z$, there's a simple way to obtain the Lagrangian projection $\pi_{xy}(T')$ of a Legendrian tangle T', which is Legendrian isotopic to T. This is realized by the resolution construction [Ng03, Prop.2.2], via a resolution procedure as in FIGURE 1.1. We say that $T' = Res(T)$ is obtained from T by *resolution construction*.

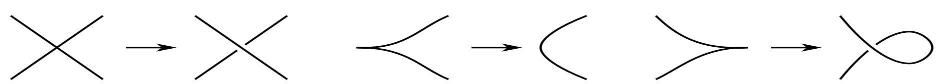

FIGURE 1.1. Resolving a front into the Lagrangian projection of a Legendrian isotopic link/tangle.

1.1.4. *Legendrian Reidemeister moves.* As in smooth knot theory, there're analogues of Reidemeister moves for Legendrian tangles via front diagrams. That is, 2 front diagrams in $U\times\mathbb{R}_z$ represent Legendrian isotopic tangles in J^1U if and only if they differ by a finite sequence of smooth isotopies and the following 3 types of *Legendrian Reidemeister moves* ([Świ92]):

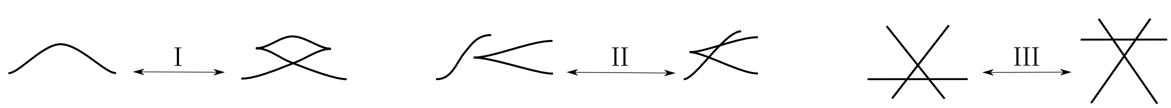

FIGURE 1.2. The 3 types of Legendrian Reidemeister moves relating Legendrian-isotopic fronts. Reflections of these moves along the coordinate axes are also allowed.

1.1.5. *Maslov potentials.* Given a Legendrian tangle T with front diagram $\pi_{xz}(T)$. Let $r = |r(T)|$ be the gcd of the rotation numbers of the closed connected components of T and m be a nonnegative integer. A $\mathbb{Z}/m\mathbb{Z}$-valued *Maslov potential* of $\pi_{xz}(T)$ is a map

$$\mu : \{\text{strands of } \pi_{xz}(T)\} \to \mathbb{Z}/m\mathbb{Z}$$

such that near any cusp, have $\mu(\text{upper strand}) = \mu(\text{lower strand}) + 1$. Such a Maslov potential exists if and only if $2r$ is a multiple of m. We will always fix a $\mathbb{Z}/2r$-valued Maslov potential

μ for T, in this case T is naturally oriented[2] by the condition that, for each strand of T, it's right-moving (resp. left-moving) if and only if μ takes even (resp. odd) value on the strand.

1.2. **Normal rulings and ruling polynomials.** Here we review the normal rulings and ruling polynomials for Legendrian tangles, following [Su17]. Given a Legendrian tangle T, with $\mathbb{Z}/2r$-valued Maslov potential μ for some fixed $r \geq 0$. Fix a nonnegative integer m dividing $2r$.

Assume that the numbers n_L, n_R of the left endpoints and right endpoints of T are both *even*. For example, any Legendrian tangle obtained from cutting a Legendrian link front along 2 vertical lines, satisfies this assumption.

Recall that the front diagram $\pi_{xz}(T)$ is divided into arcs, crossings and cusps. For example, an arc begins at a cusp, a crossing or an end-point, going from left to right, and ends at another cusp, crossing or end-point, meeting no cusp or crossing in-between. Given a crossing a of the front T, its degree is given by $|a| := \mu(\text{over-strand}) - \mu(\text{under-strand})$.

Definition 1.1. We say an embedded (closed) disk of $\overline{U} \times \mathbb{R}_z$, is an *eye* of the front T, if it is the union of (the closures of) some *regions* (See Section 1.1.2), such that the boundary of the disk in $U \times \mathbb{R}_z$, being the union of arcs, crossings and cusps, consists of 2 *paths*, starting at the same left cusp or a pair of left end-points, going from left to right through arcs and crossings, meeting no cusps in-between, and ending at the same right cusp or a pair of right end-points.

Definition 1.2. A *m-graded normal ruling* ρ of (T,μ) is a *partition* of the set of arcs of the front T into the boundaries in $U \times \mathbb{R}_z$ of *eyes* (say $e_1, \ldots, e_n$), or in other words,

$$\sqcup \text{ arcs of } T = \sqcup_{i=1}^{n} (\partial e_i \setminus \{\text{crossings, cusps}\}) \cap U \times \mathbb{R}_z,$$

and such that the following conditions are satisfied:

(1). If some eye e_i starts at a pair of left end-points (resp. ends at a pair of right end-points), we require $\mu(\text{upper-end-point}) = \mu(\text{lower-end-point}) + 1 (\text{mod} m)$.
(2). Call a crossing a a *switch*, if it's contained in the boundary of some eye e_i. In this case, we require $|a| = 0 (\text{mod} m)$.
(3). Each switch a is clearly contained in exactly 2 eyes, say e_i, e_j. We require the relative positions of e_i, e_j near a to be in one of the 3 situations in Figure 1.3(top row).

Definition 1.3. Given a Legendrian tangle (T,μ), let ρ be a m-graded normal ruling of (T,μ), and let a be a crossing. Then, a is called a *return* if the behavior of ρ at a is as in Figure 1.3(bottom row). a is called a *departure* if the behavior of ρ at a looks like one of the three pictures obtained by reflecting each of $(R1) - (R3)$ in Figure 1.3(bottom row) with respect to a vertical axis. Moreover, returns (resp. departures) of degree 0 modulo m are called *m-graded returns* (resp. *m-graded departures*) of ρ.

Define $s(\rho)$ (resp. $d(\rho)$) to be the number of switches (resp. m-graded departures) of ρ.
Define $r(\rho)$ to be the number of m-graded returns of ρ if $m \neq 1$, and the number of m-graded returns and right cusps if $m = 1$.

Definition 1.4. Given a m-graded normal ruling ρ of a Legendrian tangle (T,μ), denote by $e_1, \ldots, e_n$ the eyes in $J^1(\overline{U})$ defined by ρ. The *filling surface* S_ρ of ρ is the the disjoint union

[2] Throughout the context, Legendrian tangles will be assumed to be oriented.

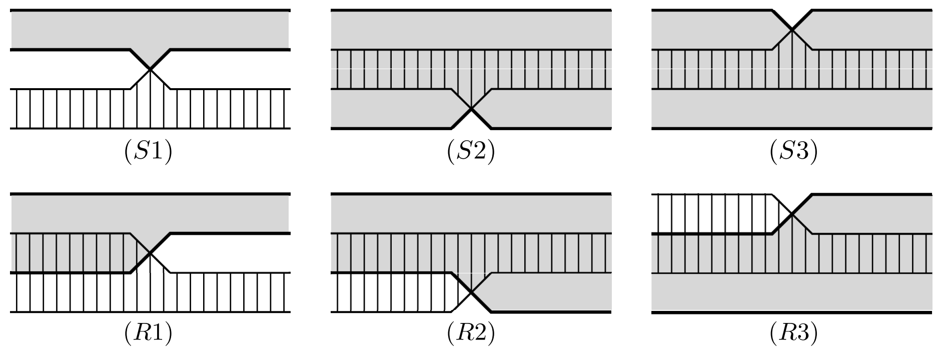

FIGURE 1.3. Top row: The behavior (of the 2 eyes e_i, e_j) of a m-graded normal ruling ρ at a *switch* (where e_i and e_j are the dashed and shadowed regions respectively), where the crossings are required to have degree 0 (mod$m$). Bottom row: The behavior (of the 2 eyes e_i, e_j) of ρ at a return. Three more figures omitted: The 3 types of departures obtained by reflecting each of (R1)-(R3) with respect to a vertical axis.

$\sqcup_{i=1}^{n} e_i$ of the eyes, glued along the switches via half-twisted strips. This is a compact surface possibly with boundary. See FIGURE 1.4 for an example.

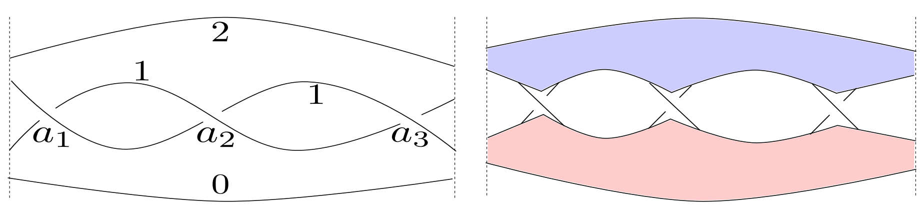

FIGURE 1.4. Left: a Legendrian tangle front T with 3 crossings a_1, a_2, a_3, the numbers indicate the values of the Maslov potential μ on each of the 4 strands. Right: the filling surface for a normal ruling of T by gluing the 2 eyes along the 3 switches via half-twisted strips.

Let T_L (resp. T_R) be the left (resp. right) pieces T near the left (resp. right) boundary. It's clear that any m-graded normal ruling ρ of T restricts to a m-graded normal ruling of the left piece T_L (resp. of the right piece T_R), denoted by $r_L(\rho)$ *or* $\rho|_{T_L}$ *(resp.* $r_R(\rho)$ *or* $\rho|_{T_R}$*).*

Definition 1.5. Fix a m-graded normal ruling ρ_L (resp. ρ_R) of T_L (resp. T_R). We define a Laurent polynomial $<\rho_L|R^m_{T,\mu}(z)|\rho_R>=<\rho_L|R^m_T(z)|\rho_R>$ in $\mathbb{Z}[z,z^{-1}]$ by

$$<\rho_L|R^m_T(z)|\rho_R>:=\sum_{\rho:\, r_L(\rho)=\rho_L,\, r_R(\rho)=\rho_R} z^{-\chi(\rho)} \tag{1.2.1}$$

where the sum is over all m-graded normal rulings ρ such that $r_L(\rho)=\rho_L, r_R(\rho)=\rho_R$. $\chi(\rho)$ is called the *Euler characteristic* of ρ and defined by

$$\chi(\rho):=\chi(S_\rho)-\chi(S_\rho|_{x=x_R}). \tag{1.2.2}$$

where x_R is the right endpoint of the open interval $U = (x_L, x_R)$ and $\chi(S_\rho)$ (resp. $\chi(S_\rho|_{x=x_R})$) is the usual Euler characteristic of S_ρ (resp. $S_\rho|_{x=x_R}$). Equivalently, $\chi(\rho) = \chi_c(S_\rho|_{x_L\leq x<x_R})$ is the Euler characteristic with compact support of $S_\rho|_{x_L\leq x<x_R}$. Also, notice that when $x_R = \infty$, $S_\rho|_{x=x_R}$ is empty with vanishing Euler characteristic.

We will call $<\rho_L|R^m_T(z)|\rho_R>$ the *m-graded ruling polynomial* of T with boundary conditions (ρ_L, ρ_R).

Remark 1.6. Given a m-graded normal ruling ρ, with $n_L = 2n'_L$ (resp. $n_R = 2n'_R$) left (resp. right) end-points and c_L (resp. c_R) left (resp. right) cusps, then $S_\rho|_{x=x_R}$ is the disjoint union of n'_R closed line segments and $n = n'_L + c_L = n'_R + c_R$ is the number of eyes in ρ. Hence, $\chi(S_\rho|_{x=x_R}) = n'_R$ is independent of ρ and we get a simple computation formula

$$\chi(\rho) = c_R - s(\rho) \tag{1.2.3}$$

where $s(\rho)$ is defined in Definition 1.3. In particular, when T is a Legendrian link, the definition here coincides with the usual definition [HR15] of ruling polynomials for Legendrian links.

Given a Legendrian tangle T, let's denote by NR^m_T (resp. $\mathrm{NR}^m_T(\rho_L, \rho_R)$) the set of m-graded normal rulings of T (resp. those with boundary conditions (ρ_L, ρ_R)). Then we have

Lemma 1.7. *[Su17, Lem.2.9] Given a Legendrian isotopy h between 2 Legendrian tangles T, T', preserving the Maslov potentials μ, μ', there's a canonical bijection between the set of m-graded normal rulings of T and T'*

$$\phi_h : \mathrm{NR}^m_T \xrightarrow{\sim} \mathrm{NR}^m_{T'}$$

commuting with the restrictions r_L, r_R, and such that for any m-graded normal ruling ρ, S_ρ and $\phi(S_\rho)$ are homeomorphic, relative to the boundary pieces at $x = x_L$ and $x = x_R$.

Note that for such 2 Legendrian isotopic tangles $(T, \mu), (T', \mu')$, their left and right pieces are necessarily identical: $T_L = T'_L, T_R = T'_R$.

As a consequence of Lemma 1.7, we have

Theorem 1.8. *[Su17, Thm.2.10] The m-graded ruling polynomials $<\rho_L|R^m_T(z)|\rho_R>$ are Legendrian isotopy invariants for (T, μ).*
Moreover, suppose $T = T_1 \circ T_2$ is the composition of two Legendrian tangles T_1, T_2, that is, $(T_1)_R = (T_2)_L$ and $T = T_1 \cup_{(T_1)_R} T_2$, then the composition axiom *for ruling polynomials holds:*

$$<\rho_L|R^m_T(z)|\rho_R> = \sum_{\rho_I} <\rho_L|R^m_{T_1}(z)|\rho_I><\rho_I|R^m_{T_2}(z)|\rho_R> \tag{1.2.4}$$

where ρ_I runs over all the m-graded normal rulings of $(T_1)_R = (T_2)_L$.

1.3. **LCH DGAs for Legendrian tangles.** Here we recall the Legendrian Contact Homology differential graded algebras (LCH DGAs) associated to any Legendrian tangles. We will follow closely the definitions in [Su17], see also [Siv11, NRS[+]15]. In the case of Legendrian knots, the LCH DGAs are naturally defined via the Lagrangian projection, which also admit a front projection description via the resolution construction [Ng03]. The LCH DGAs for Legendrian tangles are natural generalizations of those for Legendrian knots, using the front projection description.

1.3.1. *LCH DGAs via Legendrian tangle fronts.* Let (T,μ) be any Legendrian tangle (front), equipped with a $\mathbb{Z}/2r$-valued Maslov potential μ. Let $*_1,\ldots,*_B$ be the base points placed on T so that each connected component containing a right cusp has at least one base point. Suppose T has n_L (resp. n_R) left (resp. right) endpoints, labeled from top to bottom by $1,2,\ldots,n_L$ (resp. $1,2,\ldots,n_R$). Let $\{a_1,\ldots,a_R\}$ be the set of crossings and right cusps of T, let $\{a_{ij}, 1\leq i<j\leq n_L\}$ be the set of pairs of left endpoints of T.

Definition/Proposition 1.9. There's a $\mathbb{Z}/2r$-graded LCH DGA $\mathcal{A}(T)=(\mathcal{A}(T,\mu,*_1,\ldots,*_B),\partial)$ with $\deg(\partial)=-1$ as follows:

As an algebra: $\mathcal{A}(T)=\mathbb{Z}[t_1^{\pm1},\ldots,t_B^{\pm1}]<a_i, 1\leq i\leq R, a_{ij}, 1\leq i<j\leq n_L>$ is a free associative algebra over $\mathbb{Z}[t_1^{\pm1},\ldots,t_B^{\pm1}]$, where t_i is the generator corresponding to the i-th base point $*_i$, for $1\leq i\leq B$.

The grading: $|t_i^{\pm1}|=0, |a_i|=\mu(\text{over-strand})-\mu(\text{under-strand})$ if a_i is a crossing, $|a_i|=1$ if a_i is a right cusp, and $|a_{ij}|=\mu(i)-\mu(j)-1$.

The differential: We impose the graded Leibniz Rule $\partial(x\cdot y)=(\partial x)\cdot y+(-1)^{|x|}x\cdot\partial y$. It then suffices to define the differentials of the generators. These are defined as follows: $\partial(t_i^{\pm1})$=0; The differential of a_{ij}'s are given by

$$\partial a_{ij}=\sum_{i<k<j}(-1)^{|a_{ik}|+1}a_{ik}a_{kj}. \tag{1.3.1}$$

To define the differential of a crossing or a right cusp. Let $a=a_i$ and $v_1,\ldots,v_n$ be some elements in the generators $\{a_i, 1\leq i\leq R, a_{ij}, 1\leq i<j\leq n_L\}$ of T for some $n\geq 0$. Let $D_n^2=D^2-\{p,q_1,\ldots,q_n\}$ be a fixed oriented disk with $n+1$ boundary punctures (or vertices) $p,q_1,\ldots,q_n$, arranged in a counterclockwise order.

Definition 1.10. Define the moduli space $\Delta(a;v_1,\ldots,v_n)$ to be the space of *admissible disks u* of the tangle front T up to re-parametrization, that is,

(i) *(Immersion with singularities)* The map $u:(D_n^2,\partial D_n^2)\to(\mathbb{R}^2_{xz},T)$ is an immersion, orientation-preserving, and smooth away from possible singularities at left and right cusps, near which the image of the map are indicated as in FIGURE 1.5.a,b. Note that the singularities are not vertices of D_n^2;

(ii) *(Initial/positive vertex)* u extends continuously to p, with $u(p)=a$, near which the image of the map is indicated as in Figure 1.5.c;

(iii) *(Negative vertices at a crossing)* If v_i is a crossing, u extends continuously to q_i, with $u(q_i)=v_i$, near which the image of the map is indicated as in Figure 1.5.d;

(iv) *(Negative vertices at a right cusp)* If v_i is a right cusp, u extends continuously to q_i, with $u(q_i)=v_i$, near which the image of the map is indicated as in Figure 1.5.e;

(v) *(Negative vertices at a pair of left end-points)* If v_i is a pair of left end-points a_{jk}, we require that, as one approaches q_i in D_n^2, u limits to the line segment $[j,k]$ at the left boundary between the left end-points j,k of T;

(vi) The x-coordinate on the image $\overline{u(D_n^2)}$ has a unique local maximum at a.

Note: the last condition (vi) is in fact a consequence of the previous ones (i)-(v). In the case when T is a Legendrian knot (front), all the defining conditions are translated from the definition of the LCH DGA associated to $Res(T)$, via the Lagrangian projection description. Via the

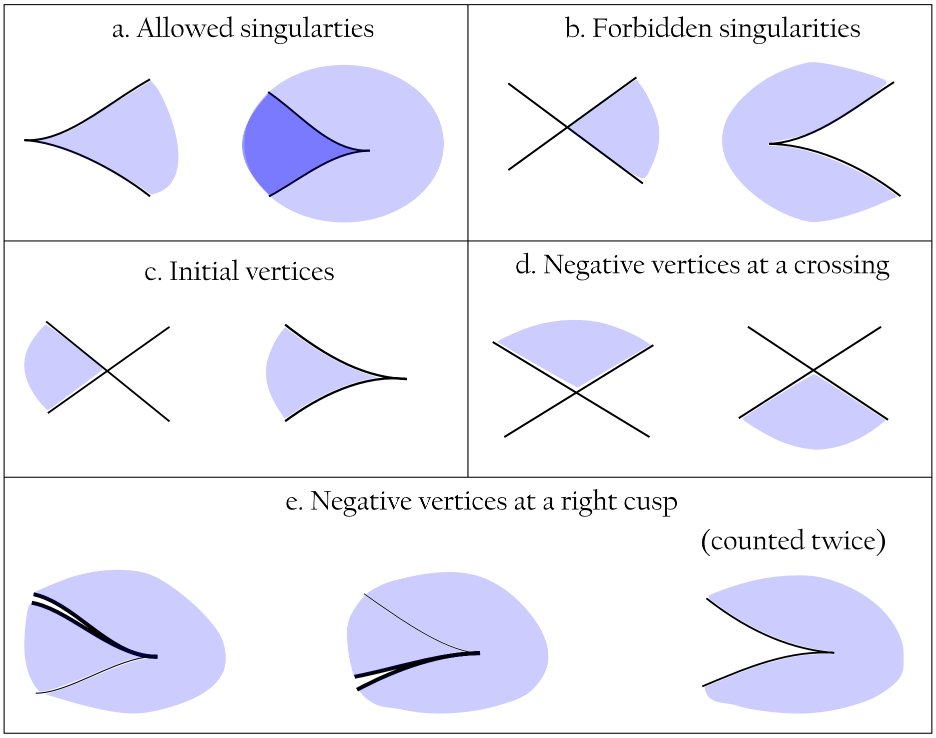

FIGURE 1.5. Admissible disks: The image of the disk D_n^2 under an admissible map near a singularity or a vertex on the boundary ∂D_n^2. The first row indicates the possible singularities, the second and third rows indicate the possible vertices. In the first 2 pictures of part e, 2 copies of the same strand (the heavy lines) are drawn for clarity.

resolution construction (Figure 1.1), the defining conditions near a right cusp are illustrated by Figure 1.6.

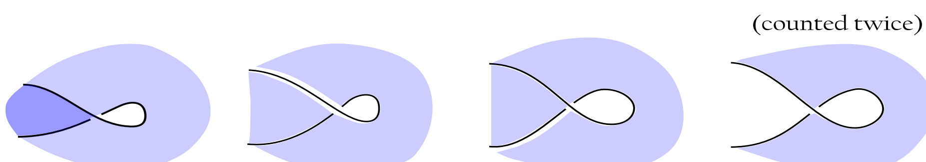

FIGURE 1.6. The singularity and negative vertices at a right cusp after resolution: The first figure corresponds to a singularity (Figure 1.5.a), the remaining ones correspond to a negative vertex (FIGURE 1.5.e, going from left to right).

For each $u \in \Delta(a; v_1, \ldots, v_n)$, walk along $\overline{u(\partial D_n^2)}$ starting from a in counterclockwise direction, we encounter a sequence $s_1, \ldots, s_N (N \geq n)$ of negative vertices of u (crossings, right cusps, or pairs of left end-points as in Definition 1.10) and base points (away from the previous negative vertices).

Definition 1.11. The *weight* of u is $w(u) := w(s_1)\dots w(s_N)$, where

(i) $w(s_k) = t_i$(resp. t_i^{-1}) if s_k is the base point $*_i$, and the boundary orientation of $u(\partial D^2)$ agrees (resp. disagrees) with the orientation of T near $*_i$. Note that this includes the case when the base point $*_i$ is located at a right cusp, which is also a singularity of u (See Figure 1.5.a);

(ii) $w(s_k) = v_i$ (resp. $(-1)^{|v_i|+1}v_i$) if s_k is the crossing v_i and the disk $\overline{u(D_n^2)}$ occupies the top (resp. bottom) quadrant of v_i (See Figure 1.5.d);

(iii) $w(s_k) = a_{ij}$ if s_k is the pair of left end-points a_{ij};

(iv) $w(s_k) = w_1(s_k)w_2(s_k)$ if s_k is the right cusp $v_i = u(q_i)$ (see Figure 1.5.e), where
$w_2(s_k) = v_i$ (resp. v_i^2) if the image of u near q_i looks like the first two diagrams (resp. the third diagram) of Figure 1.5.e;
$w_1(s_k) = 1$ if s_k is a unmarked right cusp (equipped with no base point);
$w_1(s_k) = t_j$ (resp. t_j^{-1}) if v_i is a marked right cusp equipped with the base point $*_j$, and v_i is an up (resp. down) right cusp[3]. See Figure 1.6 for an illustration.

Definition 1.12. For $a = a_i$ a crossing or a right cusp, its differential is given by

$$\partial a = \sum_{n,v_1,\dots,v_n} \sum_{u\in\Delta(a;v_1,\dots,v_n)} w(u) \tag{1.3.2}$$

where for $a = a_i$ a right cusp, we also include the contribution from an "invisible" disk u coming from the resolution construction (see Figure 1.1 (right)), with $w(u) = 1$ (resp. t_j^{-1} or t_j), if there's no base point (resp. a base point $*_j$, depending on whether a_i is an up or down right cusp).

1.3.2. *The co-sheaf property.* Let T be a Legendrian tangle in J^1U. Let V be an open subinterval of U such that, the boundary $(\partial\overline{U})\times\mathbb{R}_z$ is disjoint from the crossings, cusps and base points of T. $T|_V$ then gives a Legendrian tangle in J^1V with Maslov potential induced from that of T, hence the LCH DGA $\mathcal{A}(T|_V)$ is defined. There's indeed a co-restriction map of DGAs.

Definition/Proposition 1.13 ([NRS+15, Prop.6.12], [Siv11] or [Su17, Def/Prop.3.9])**.** The following defines a morphism of $\mathbb{Z}/2r$-graded DGAs $\iota_{UV} : \mathcal{A}(T|_V) \to \mathcal{A}(T)$:

(1) ι_{UV} sends a generator of $\mathcal{A}(T|_V)$, corresponding to a crossing, a right cusp or a base point of T, to the corresponding generator of $\mathcal{A}(T)$;

(2) For a generator b_{ij} in $\mathcal{A}(T|_V)$ corresponding to the pair of left end-points i, j of $T|_V$, the image $\iota_{UV}(b_{ij})$ is defined as follows:
Use the notations in Definition 1.10 and consider the moduli space $\Delta(b_{ij}; v_1,\dots,v_n)$ of disks $u : (D_n^2, \partial D_n^2) \to (\mathbb{R}^2_{xz}, T)$ satisfying the conditions in definition 1.10, with the condition for a there replaced by "u limits to the line segments $[i, j]$ between the pair of left end-points i, j of $T|_V$ at the puncture $p \in \partial D^2$ and u attains its local maxima exactly along $[i, j]$". Then define

$$\iota_{UV}(b_{ij}) = \sum_{n,v_1,\dots,v_n} \sum_{u\in\Delta(b_{ij},v_1,\dots,v_n)} w(u) \tag{1.3.3}$$

[3]Recall that a cusp is called *up* (resp. *down*) if the orientation of the front T near the cusp goes up (resp. down). See Figure 1.8.

Example 1.14 (co-restriction ι_R for a right cusp)**.** One key example for the co-restriction of DGAs is $\iota_R : \mathcal{A}(T_R) \to \mathcal{A}(T)$, where T be an elementary Legendrian tangle of a single (marked or unmarked) right cusp a, and T_R is the right piece of T. For simplicity, assume T has 4 left endpoints and 2 right endpoints as in Figure 1.7. Then $\mathcal{A}(T_R) = \mathbb{Z} < b_{12} >$, where b_{12} is the generator corresponding to the pair of left endpoints of T_R, and $\mathcal{A}(T) = \mathbb{Z}[t, t^{-1}] < a, a_{ij}, 1 \leq i < j \leq 4 >$ with $\partial a = t^{\sigma(a)} + a_{23}$ (see Definition 1.15 below), where a_{ij}'s correspond to the pairs of left endpoints of T, t is the generator corresponding to the base point if the right cusp is marked and $t = 1$ otherwise. Then $\iota_R : \mathcal{A}(T_R) \to \mathcal{A}(T)$ is given by

$$\begin{aligned}\iota_R(b_{12}) &= a_{14} + a_{13}t^{-\sigma(a)}a_{24} + a_{12}at^{-\sigma(a)}a_{24} + a_{13}t^{-\sigma(a)}aa_{34} + a_{12}at^{-\sigma(a)}aa_{34} \\ &= a_{14} + t^{-\sigma(a)}(a_{13} + a_{12}a)(a_{24} + aa_{34}).\end{aligned}$$

FIGURE 1.7. Left: An elementary Legendrian tangle of an unmarked right cusp. Right: An elementary Legendrian tangle of a marked right cusp.

Here, we introduce a sign at a right cusp:

Definition 1.15. Given a right cusp a of the oriented tangle front T, we define the *sign* $\sigma = \sigma(a)$ of a to be 1 (resp. -1) if a is a down (resp. up) cusp. See Figure 1.8.

FIGURE 1.8. Left: a down right cusp. Right: an up right cusp.

One key property of LCH DGAs for Legendrian tangles is the co-sheaf property:

Proposition 1.16 ([NRS+15, Thm.6.13] or [Su17, Prop.3.13])**.** *If $U = L \cup_V R$ is the union of 2 open intervals L, R with non-empty intersection V, then the diagram of co-restriction maps*

(1.3.4)
$$\begin{tikzcd} \mathcal{A}(T|_V) \arrow[r, "\iota_{RV}"] \arrow[d, "\iota_{LV}"'] & \mathcal{A}(T|_R) \arrow[d, "\iota_{UR}"] \\ \mathcal{A}(T|_L) \arrow[r, "\iota_{UL}"] & \mathcal{A}(T) \end{tikzcd}$$

gives a pushout square of $\mathbb{Z}/2r$-graded DGAs.

1.4. **Augmentations and the ruling decomposition.** In this subsection, we will review the augmentation varieties (with given boundary conditions) for Legendrian tangles, following [Su17, Section.4.1].

Fix a Legendrian tangle T, with $\mathbb{Z}/2r$-valued Maslov potential μ, base points $*_1,\ldots,*_B$ so that each connected component containing a right cusp has at least one base point. Denote the crossings, right cusps and pairs of left end-points by $\mathcal{R}=\{a_1,\ldots,a_N\}$. As always, the base points are assumed to be away from the crossings and left cusps of T. Let n_L, n_R be the numbers of left and right end-points in T respectively.

1.4.1. *Full augmentation varieties.* We define the LCH DGA $(\mathcal{A}(T),\partial)$ as in the previous Section 1.3. So as an associative algebra we have $\mathcal{A}:=\mathcal{A}(T)=\mathbb{Z}[t_1^{\pm 1},\ldots,t_B^{\pm 1}]<a_1,\ldots,a_N>$. Fix a nonnegative integer m dividing r and a base field k.

Definition 1.17. A *m-graded (or $\mathbb{Z}/m$-graded) k-augmentation* of $\mathcal{A}$ is an unital algebraic map $\epsilon:(\mathcal{A},\partial)\to(k,0)$ such that $\epsilon\circ\partial=0$, and for all a in $\mathcal{A}$ we have $\epsilon(a)=0$ if $|a|\neq 0(\mathrm{mod} m)$. Here $(k,0)$ is viewed as a DGA concentrated on degree 0 with zero differential. Morally, "ϵ is a $\mathbb{Z}/m\mathbb{Z}$-graded DGA map".

Definition 1.18. Define $\mathrm{Aug}_m(T,k)$ to be the set of m-graded k-augmentations of $\mathcal{A}(T)$. This defines an affine subvariety of $(k^\times)^B\times k^N$, via the map

$$\mathrm{Aug}_m(T,k)\ni\epsilon\to(\epsilon(t_1,\ldots,\epsilon(t_B),\epsilon(a_1),\ldots,\epsilon(a_N)))\in(k^\times)^B\times k^N$$

with the defining polynomial equations $\epsilon\circ\partial(a_i)=0, 1\leq i\leq N$ and $\epsilon(a_i)=0$ for $|a_i|\neq 0(\mathrm{mod} m)$. This affine variety $\mathrm{Aug}_m(T,k)$ will be called the *(full) m-graded augmentation variety* of $(T,\mu,*_1,\ldots,*_B)$.

Augmentation varieties for Legendrian tangles satisfy a sheaf property, induced by the co-sheaf property of LCH DGAs in Section 1.3.2. More precisely, we have

Definition/Proposition 1.19 (Sheaf property for augmentation varieties)**.** Let T a Legendrian tangle in J^1U.

(1) Let V be an open subinterval of U, then the co-restriction of DGAs $\iota_{UV}:\mathcal{A}(T|_V)\to\mathcal{A}(T)$ induces a restriction $r_{VU}=\iota_{VU}^*:\mathrm{Aug}_m(T;k)\to\mathrm{Aug}_m(T|_V;k)$.
(2) If $U=L\cup_V R$ is the union of 2 open intervals L,R with non-empty intersection V, then the diagram of restriction maps

$$\begin{tikzcd} \mathrm{Aug}_m(T;k) \arrow[r, "r_{RU}"] \arrow[d, "r_{LU}"'] & \mathrm{Aug}_m(T|_R;k) \arrow[d, "r_{VR}"] \\ \mathrm{Aug}_m(T|_L;k) \arrow[r, "r_{VL}"] & \mathrm{Aug}_m(T|_V;k) \end{tikzcd} \tag{1.4.1}$$

gives a fiber product of augmentation varieties.

1.4.2. *Barannikov normal forms.*

Example 1.20 (The augmentation variety for trivial Legendrian tangles)**.** Let T be the trivial Legendrian tangle of n parallel strands, labeled from top to bottom by $1,2,\ldots,n$, equipped a $\mathbb{Z}/2r$-valued Maslov potential μ. The LCH DGA is $\mathcal{A}(T)=\mathbb{Z}<a_{ij}, 1\leq i<j\leq n>$, with the grading $|a_{ij}| = \mu(i) - \mu(j) - 1$ and the differential given by formula (1.3.1). The m-graded augmentation variety $\mathrm{Aug}_m(T;k)$ is

$$\mathrm{Aug}_m(T;k) = \{(\epsilon(a_{ij}))_{1\leq i<j\leq n} | \epsilon \circ \partial a_{ij} = 0, \text{ and } \epsilon(a_{ij}) = 0 \text{ if } |a_{ij}| \neq 0 (\mathrm{mod} m).\}$$

On the other hand,

Definition 1.21. Associate to the trivial Legendrian tangle (T,μ), *define* a canonical $\mathbb{Z}/m$-graded filtered k-module $C = C(T)$: C is the free k-module generated by $e_1,\ldots,e_n$ corresponding to the n strands of T with grading $|e_i| = \mu(i)(\mathrm{mod} m)$. Moreover, C is equipped with a decreasing filtration $F^0 \supset F^1 \supset \ldots \supset F^n : F^iC = \mathrm{Span}\{e_{i+1},\ldots,e_n\}$.
Define $B_m(T) := \mathrm{Aut}(C)$ to be the automorphism group of the $\mathbb{Z}/m$-graded filtered k-module C. *Denote* $I = I(T) := \{1,2,\ldots,n\}$.

Now, in the example, given any m-graded augmentation ϵ for $\mathcal{A}(T)$, we construct a $\mathbb{Z}/m$-graded chain complex $C(\epsilon) = (C, d(\epsilon))$: The differential $d = d(\epsilon)$ is filtration preserving, of degree -1 given by

$$< de_i, e_j >= 0 \text{ for } i \geq j \text{ and } < de_i, e_j >= (-1)^{\mu(i)}\epsilon(a_{ij}) \text{ for } i < j.$$

Here $< de_i, e_j >$ denotes the coefficient of e_j in de_i. The condition that d is of degree -1 is equivalent to: $< de_i, e_j >= (-1)^{\mu(i)}\epsilon(a_{ij}) = 0$ if $\mu(i) - \mu(j) - 1 = |a_{ij}| \neq 0(\mathrm{mod} m)$ for all $i < j$. The condition of the differential $d^2 = 0$ is equivalent to: for all $i < j$ have $< d^2e_i, e_j >= \sum_{i<k<j} < de_i, e_k >< de_k, e_j >= 0$, i.e. $\sum_{i<k<j}(-1)^{\mu(i)-\mu(k)}\epsilon(a_{ik})\epsilon(a_{kj}) = \epsilon \circ \partial a_{ij} = 0$.

Thus, we see that the map $\epsilon \to C(\epsilon)$ gives an *isomorphism* between the augmentation variety $\mathrm{Aug}_m(T;k)$ and the set $MCS_m^A(T;k)$ of $\mathbb{Z}/m$-graded filtered chain complexes (C,d), or equivalently, the set of filtration preserving degree -1 differentials d of C. From now on, we will always use this identification (see also Section 3.1).

Given the Legendrian tangle (T,μ) of n parallel strands, $B_m(T)$ acts naturally on $\mathrm{Aug}_m(T;k) = MCS_m^A(T)$ via conjugation: given $\varphi \in B_m(T)$ and (C,d) in $MCS_m^A(T;k)$, have $\varphi\cdot(C,d) := (C, \varphi\circ d\circ\varphi^{-1})$. In particular, the $B_m(T)$-orbit $B_m(T)\cdot(C,d)$ (or $B_m(T)\cdot d$) is simply the isomorphism classes of d.

Lemma 1.22 (Barannikov normal form, [Su17, Lem.4.5], or [Bar94, Lau15])**.** *Let (C,d) be any $\mathbb{Z}/m$-graded filtered chain complex over k, where $C = \mathrm{Span}_k\{e_1,\ldots,e_n\}$ is fixed with the decreasing filtration $F^0 \supset F^1 \supset \ldots \supset F^n$: $F^iC = \mathrm{Span}_k\{e_{i+1},\ldots,e_n\}$, then the isomorphism class of (C,d) has a unique representative, say (C,d_0), such that the matrix $(< d_0e_i, e_j >)_{i,j}$ has at most one nonzero entry in each row and column and moreover these are all 1's. Equivalently, for $I = \{1,2,\ldots,n\}$, there's a partition $I = U \sqcup L \sqcup H$ and a bijection $\rho : U \xrightarrow{\sim} L$, satisfying $\rho(i) > i$ and $|e_i| = |e_{\rho(i)}| + 1(\mathrm{mod} m)$ for all i in U, and such that $d_0e_i = e_{\rho(i)}$ for $i \in U$, $d_0e_i = 0$ for $i \in L \sqcup H$.*

The unique representative (C,d_0) is called the Barannikov normal form *of (C,d).*

Definition 1.23. Given a trivial Legendrian tangle (T,μ), a partition $I = I(T) = U \sqcup L \sqcup H$ together with a bijection $\rho : U \xrightarrow{\sim} L$ as in Lemma 1.22, will be called an *m-graded isomorphism type* of T, denoted by ρ for simplicity.

Remark 1.24. By Lemma 1.22, each m-graded isomorphism type ρ of T determines an unique isomorphism class $O_m(\rho; k)$ of $\mathbb{Z}/m$-graded filtered k-complexes $(C(T), d)$. In other words, $O_m(\rho; k)$ is the $B_m(T)$-orbit of the *canonical augmentation* ϵ_ρ (equivalently, the *Barannikov normal form* d_ρ determined by ρ), using the identification in Example 1.20. We thus obtain a decomposition of the full augmentation variety for the trivial Legendrian tangle (T, μ):

$$\mathrm{Aug}_m(T; k) = \sqcup_\rho O_m(\rho; k) \tag{1.4.2}$$

where ρ runs over all m-graded isomorphism types of T.

In addition, take a m-graded augmentation ϵ of $\mathcal{A}(T)$, or equivalently the m-graded filtered chain complex $C(\epsilon) = (C, d(\epsilon))$. Suppose ϵ is *acyclic*, meaning that $(C, d(\epsilon))$ is acyclic, i.e. $H = \emptyset$ in the partition $I = L \sqcup H \sqcup U$ associated to $d(\epsilon)$. Then, the associated m-graded isomorphism type $\rho : U \xrightarrow{\sim} L$ can be identified with an *m-graded normal ruling* (denoted by the same ρ) of T.

1.4.3. *Augmentation varieties with fixed boundary conditions.* Now, we come back to the general case. Let $(T, \mu, *_1, \ldots, *_B)$ be any Legendrian tangle as in the beginning of this subsection. Take the left and right pieces of T, called T_L, T_R respectively. We get 2 restrictions of augmentation varieties

$$r_L = \iota_L^* : \mathrm{Aug}_m(T) \to \mathrm{Aug}_m(T_L) \tag{1.4.3}$$

$$r_R = \iota_R^* : \mathrm{Aug}_m(T) \to \mathrm{Aug}_m(T_R). \tag{1.4.4}$$

By the sheaf property of augmentation varieties, it's natural to consider the following subvarieties:

Definition 1.25. Given m-graded isomorphism types ρ_L, ρ_R for T_L, T_R respectively, and $\epsilon_L \in O_m(\rho_L; k)$. Define the varieties

$$\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k) := \{\epsilon_L\} \times_{\mathrm{Aug}_m(T_L;k)} \times \mathrm{Aug}_m(T; k) \times_{\mathrm{Aug}_m(T_R;k)} \times O_m(\rho_R; k)$$
$$\mathrm{Aug}_m(T, \rho_L, \rho_R; k) := O_m(\rho_L; k) \times_{\mathrm{Aug}_m(T_L;k)} \times \mathrm{Aug}_m(T; k) \times_{\mathrm{Aug}_m(T_R;k)} \times O_m(\rho_R; k)$$

$\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k)$ will be called the *m-graded augmentation variety with boundary conditions* (ϵ_L, ρ_R) for T. When $\epsilon_L = \epsilon_{\rho_L}$ is the *canonical augmentation* of T_L corresponding to the Barannikov normal form determined by ρ_L, we will call $\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; k)$ the *m-graded augmentation variety (with boundary conditions* (ρ_L, ρ_R)*)* of T.

By definition, we immediately obtain a decomposition of the full augmentation variety

$$\mathrm{Aug}_m(T; k) = \sqcup_{\rho_L, \rho_R} \mathrm{Aug}_m(T, \rho_L, \rho_R; k) \tag{1.4.5}$$

where ρ_L, ρ_R run over all m-graded isomorphism types of T_L, T_R respectively.

From now on, we will *consider only* the varieties $\mathrm{Aug}_m(T, \rho_L, \rho_R; k)$ (or $\mathrm{Aug}_m(T, \epsilon_L, \rho_R; k)$ for some $\epsilon_L \in O_m(\rho_L; k)$), where ρ_L, ρ_R are m-graded normal rulings of T_L, T_R respectively. In particular, this *forces* that the numbers n_L, n_R of left endpoints and right endpoints of T are both *even*.

Definition 1.26. Let $\mathbb{F}_q$ be any finite field, and ρ_L, ρ_R be *m-graded normal rulings* of T_L, T_R respectively. The *m-graded augmentation number (with boundary conditions* (ρ_L, ρ_R)*)* of T

over $\mathbb{F}_q$ is

$$\mathrm{aug}_m(T,\rho_L,\rho_R;q) := q^{-\dim_{\mathbb{C}}\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;\mathbb{C})}|\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;\mathbb{F}_q)| \tag{1.4.6}$$

where $|\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;\mathbb{F}_q)|$ is simply the counting of $\mathbb{F}_q$-points.

Augmentation numbers are invariants computed by ruling polynomials:

Theorem 1.27 ([Su17, Thm.4.19]). *Let (T,μ) be a Legendrian tangle, with B base points so that each connected component containing a right cusp has at least one base point. Fix $m|2r$ and m-graded normal rulings ρ_L,ρ_R of T_L,T_R respectively, then*

$$\mathrm{aug}_m(T,\rho_L,\rho_R;q) = q^{-\frac{d+B}{2}}z^B < \rho_L|R^m_T(z)|\rho_R >$$

where $q=|\mathbb{F}_q|$, $z=q^{\frac{1}{2}}-q^{-\frac{1}{2}}$, and d is the maximal degree in z of $< \rho_L|R^m_T(z)|\rho_R >$.

In other words, the point-counting, or equivalently by [HRV08, Katz's appendix], the weight polynomials of the augmentation varieties $\mathrm{Aug}_m(T,\epsilon_{\rho_L},\rho_R;k)$, recover the ruling polynomials.

1.4.4. *The ruling decomposition.* Given a Legendrian tangle (T,μ). *Assume T* is placed with B base points so that *each right cusp is marked*. Label the crossings, cusps and base points away from the right cusps of T by $q_1,\ldots,q_n$ with x-coordinates, from left to right. Let $x_0 < x_1 < \ldots < x_n$ be the x-coordinates which cut T into elementary tangles. That is, x_0 and x_n are the the x-coordinates of the left and right end-points of T, and $x_{i-1} < x_{q_i} < x_i$ for all $1\le i\le n$. Let $T_i = T|_{\{x_0<x<x_i\}}$ and $E_i := T|_{\{x_{i-1}<x<x_i\}}$ be the i-th elementary tangle around q_i, then $T=T_n=E_1\circ E_2\circ\ldots\circ E_n$ is the *composition* of n elementary tangles.

Fix m-graded normal rulings ρ_L,ρ_R of T_L,T_R respectively. Fix $\epsilon_L\in\mathcal{O}_m(\rho_L;k)$.

Definition 1.28. For any m-graded normal ruling ρ of T such that $\rho|_{T_L}=\rho_L$ and $\rho|_{T_R}=\rho_R$, denote $\rho_i := \rho|_{(T_i)_R=(T_{i+1})_L}$ for $0\le i\le n$. In particular, $\rho_0=\rho_L,\rho_n=\rho_R$. Use the left and right restriction maps for augmentation varieties, to define the varieties

$$\begin{aligned}
\mathrm{Aug}^{\rho}_m(T,\epsilon_L) &:= \mathrm{Aug}_m(E_1,\epsilon_L,\rho_1)\times_{\mathcal{O}_m(\rho_1)}\ldots\times_{\mathcal{O}_m(\rho_{n-1})}\mathrm{Aug}_m(E_n,\rho_{n-1},\rho_n)\\
\mathrm{Aug}^{\rho}_m(T,\rho_L) &:= \mathrm{Aug}_m(E_1,\rho_0,\rho_1)\times_{\mathcal{O}_m(\rho_1)}\ldots\times_{\mathcal{O}_m(\rho_{n-1})}\mathrm{Aug}_m(E_n,\rho_{n-1},\rho_n)
\end{aligned}$$

while for simplicity we have ignored the coefficient field k. Sometimes for clarity, we also denote $\mathrm{Aug}^{\rho}_m(T,\epsilon_L,\rho_R;k) := \mathrm{Aug}^{\rho}_m(T,\epsilon_L;k)$ and $\mathrm{Aug}^{\rho}_m(T,\rho_L,\rho_R;k) = \mathrm{Aug}^{\rho}_m(T;k) := \mathrm{Aug}^{\rho}_m(T,\rho_L;k)$.

We then have the *ruling decomposition*:

Theorem 1.29 ([Su17, Thm.5.10] or [HR15, Thm.3.4]). *Let (T,μ) be any Legendrian tangle, with B base points placed on T so that* each right cusp is marked. *Fix m-graded normal rulings ρ_L,ρ_R of T_L,T_R respectively. Fix $\epsilon_L\in\mathcal{O}_m(\rho_L;k)$. Then there's a decomposition of augmentation varieties into the disjoint union of subvarieties*

$$\mathrm{Aug}_m(T,\epsilon_L,\rho_R;k) = \sqcup_{\rho}\mathrm{Aug}^{\rho}_m(T,\epsilon_L,\rho_R;k)$$
$$\mathrm{Aug}_m(T,\rho_L,\rho_R;k) = \sqcup_{\rho}\mathrm{Aug}^{\rho}_m(T,\rho_L,\rho_R;k)$$

where ρ runs over all m-graded normal rulings of T such that $\rho|_{T_L}=\rho_L,\rho|_{T_R}=\rho_R$. Moreover,

$$\mathrm{Aug}^{\rho}_m(T,\epsilon_L,\rho_R;k) \cong (k^*)^{-\chi(\rho)+B}\times k^{r(\rho)} \tag{1.4.7}$$

$$\mathrm{Aug}^{\rho}_m(T,\rho_L,\rho_R;k) \cong \mathcal{O}_m(\rho_L;k)\times(k^*)^{-\chi(\rho)+B}\times k^{r(\rho)} \tag{1.4.8}$$

$$\cong \quad (k^*)^{-\chi(\rho)+B+n'_L} \times k^{r(\rho)+A(\rho_L)}$$

where $n_L = 2n'_L$ is the number of left endpoints of T, and $A(\rho_L)$ is defined below.

Definition 1.30. As in Definition 1.23, let ρ be any m-graded isomorphism type for a trivial Legendrian tangle T: $I = I(T) = U \sqcup L \sqcup H, \rho : U \xrightarrow{\sim} L$. For any $i \in I$, define

$$I(i) := \{j \in I | j > i, \mu(j) = \mu(i) (\mathrm{mod} m)\}.$$

For any $i \in U \sqcup H$, define

$$A(i) = A_\rho(i) := \{j \in U \sqcup H | j \in I(i) \text{ and } \rho(j) < \rho(i)\}.$$

where for $i \in H$, denote $\rho(i) := \infty$. Now, define $A(\rho) \in \mathbb{N}$ by

$$A(\rho) := \sum_{i \in U \sqcup H} |A(i)| + \sum_{i \in L} |I(i)|.$$

From now on, we will always *assume* that *each right cusp of a Legendrian tangle is marked*.

Remark 1.31. By [Su17, Lem.4.20], the index $-\chi(\rho) + 2r(\rho)$ only depends on $T, \rho|_{T_L}, \rho|_{T_R}$. Hence, so is $a(\rho) + 2b(\rho)$, where $a(\rho) = -\chi(\rho) + B + n'_L, b(\rho) = r(\rho) + A(\rho_L)$.

Remark 1.32. By the previous theorem, we obtain a natural surjection

$$R_T : \mathrm{Aug}_m(T, \rho_L, \rho_R; k) \to \mathrm{NR}^m_T(\rho_L, \rho_R)$$

which sends an augmentation to its underlying m-graded normal ruling.

2. On the cohomology of the augmentation varieties

Let (T, μ) be an oriented Legendrian tangle. Given any augmentation variety with fixed boundary conditions associated to (T, μ), the mixed Hodge structure on its compactly supported cohomology, up to a normalization, is a Legendrian isotopy invariant (Section 3). In this section, associated to the ruling decomposition of the variety, we derive a spectral sequence converging to the mixed Hodge structure. As an application, we obtain some knowledge on the cohomology of the augmentation variety.

2.1. A spectral sequence converging to the mixed Hodge structure.

As in Section 1.4.4, let (T, μ) be an oriented Legendrian tangle with each right cusp marked, and $T = E_1 \circ E_2 \circ \ldots E_n$ is the composition of n elementary Legendrian tangles. Fix m-graded normal rulings ρ_L, ρ_R of T_L, T_R respectively. Denote by $\mathrm{NR}^m_T(\rho_L, \rho_R)$ the set of m-graded normal rulings ρ of T such that $\rho|_{T_L} = \rho_L, \rho|_{T_R} = \rho_R$.

For each $1 \leq i \leq n-1$, recall that the co-restriction of LCH DGAs induce a restriction map of augmentation varieties $r_i : \mathrm{Aug}_m(T, \rho_L, \rho_R; k) \to \mathrm{Aug}^a_m((E_i)_R = (E_{i+1})_L; k)$, where $\mathrm{Aug}^a_m((E_i)_R = (E_{i+1})_L; k)$ is the variety of acyclic augmentations (See Remark 1.24) of $(E_i)_R = (E_{i+1})_L$. Take the underlying normal rulings, r_i induces the restriction map on the sets of normal rulings $r_i : \mathrm{NR}^m_T(\rho_L, \rho_R; k) \to \mathrm{NR}^m_{(E_i)_R}$, given by $r_i(\rho) = \rho|_{(E_i)_R}$. Moreover, the ruling decomposition $\mathrm{Aug}^a_m((E_i)_R; k) = \sqcup_\tau \mathrm{Aug}^\tau_m((E_i)_R; k) = O_m(\tau; k)$ is a stratification stratified by the $B_m((E_i)_R)$-orbits, where τ runs over the set $\mathrm{NR}^m_{(E_i)_R}$ of all m-graded normal rulings of $(E_i)_R$.

Definition 2.1. Firstly, define a *geometric partial order* $\leq^G$ on $\mathrm{NR}^m_{(E_i)_R}$ via inclusions of strata: For any τ, τ' in $\mathrm{NR}^m_{(E_i)_R}$, we say $\tau' \leq^G \tau$, if $O_m(\tau'; k) \subset \overline{O_m(\tau; k)}$ in $\mathrm{Aug}^a_m((E_i)_R; k)$.

Now, define an *algebraic partial order* $\leq^A$ on $\mathrm{NR}^m_T(\rho_L, \rho_R)$: For any ρ, ρ' in $\mathrm{NR}^m_T(\rho_L, \rho_R)$, we say $\rho' \leq^A \rho$, if $r_i(\rho') \leq^G r_i(\rho)$ for all $1 \leq i \leq n-1$.

Definition 2.2. For each m-graded normal ruling ρ of T, define a *closed subvariety* $A_\rho(T; k)$ of $\mathrm{Aug}_m(T, \rho_L, \rho_R; k)$:

$$A_\rho(T; k) := \{\epsilon \in \mathrm{Aug}_m(T, \rho_L, \rho_R; k) | R_T(\epsilon) \leq^A \rho\}$$

Notice that $A_\rho(T; k) = \cap_{i=1}^{n-1} r_i^{-1}(\overline{O_m}(r_i(\rho); k))$, so it's indeed a closed subvariety. It's also clear that $A_\rho(T; k) = \sqcup_{\rho' \leq^A \rho} \mathrm{Aug}^{\rho'}_m(T; k)$ set-theoretically.

The ruling decomposition induces a finite *ruling filtration* of $\mathrm{Aug}_m(T, \rho_L, \rho_R; k)$ by closed subvarieties:

Definition/Proposition 2.3. Define a decomposition $\mathrm{NR}^m_T(\rho_L, \rho_R) = \sqcup_{i=0}^D R_i$ by induction: Let $D+1$ be the maximal length of the ascending chains in $(\mathrm{NR}^m_T(\rho_L, \rho_R), \leq^A)$. Let R_D is the subset of maximal elements in $(\mathrm{NR}^m_T(\rho_L, \rho_R), \leq^A)$. Suppose we've defined $R_{i+1}, \ldots, R_D$, let R_i be the subset of maximal elements in $(\mathrm{NR}^m_T(\rho_L, \rho_R) - \sqcup_{j=i+1}^D R_j, \leq^A)$.

Now, *define* the closed subvariety $A_i = A_i(T, \rho_L, \rho_R; k)$ of $\mathrm{Aug}_m(T, \rho_L, \rho_R; k)$ as

$$A_i := \cup_{\rho \in R_i} A_\rho(T; k) \tag{2.1.1}$$

for $0 \leq i \leq D$. By definition, we obtain a finite filtration:

$$\mathrm{Aug}_m(T, \rho_L, \rho_R; k) = A_D \supset A_{D-1} \supset \ldots \supset A_0 \supset A_{-1} = \emptyset \tag{2.1.2}$$

Moreover, as varieties we have

$$A_i - A_{i-1} = \sqcup_{\rho \in R_i} \mathrm{Aug}^\rho_m(T; k) \tag{2.1.3}$$

That is, $A_i - A_{i-1}$ is the disjoint union of some open subvarieties $\mathrm{Aug}^\rho_m(T; k)$.

Proof. It suffices to show the last identity. This is clear set-theoretically, it's enough to show each $\mathrm{Aug}^\rho_m(T; k)$ is an open subvariety of $A_i - A_{i-1}$. We only need to show that, for any $\rho \neq \rho'$ in R_i, have $\mathrm{Aug}^\rho_m(T; k) \cap \overline{\mathrm{Aug}^{\rho'}_m(T; k)} = \emptyset$. Otherwise, say, $\epsilon \in \mathrm{Aug}^\rho_m(T; k) \cap \overline{\mathrm{Aug}^{\rho'}_m(T; k)}$, then $R_T(\epsilon) = \rho$, and $\epsilon \in \overline{\mathrm{Aug}^{\rho'}_m(T; k)} \subset r_i^{-1}(\overline{O_m}(r_i(\rho'); k))$ for all $1 \leq i \leq n-1$. It follows that $r_i(\epsilon) \in \overline{O_m}(r_i(\rho'); k)$, hence $r_i(\rho) \leq^G r_i(\rho')$ for all $1 \leq i \leq n-1$, that is, $\rho' \leq^A \rho$. However, ρ is maximal in $\mathrm{NR}^m_T(\rho_L, \rho_R) - \sqcup_{j=i+1}^D R_j$, so $\rho = \rho'$, contradiction. □

Now, the ruling filtration induces a spectral sequence computing the mixed Hodge structure (Definition/Proposition 2.5) of the augmentation variety $A_D = \mathrm{Aug}_m(T, \rho_L, \rho_R; \mathbb{C})$:

Lemma 2.4. *Any finite filtration $A_D \supset A_{D-1} \supset \ldots \supset A_0 \supset A_{-1} = \emptyset$ by closed subvarieties induces a* spectral sequence *converging to the compactly supported cohomology of the variety A_D, respecting the mixed Hodge structures*[4] *(MHS):*

$$E_1^{p,q} = H_c^{p+q}(A_p \setminus A_{p-1}) \Rightarrow H_c^{p+q}(A_D).$$

[4]For simplicity, we will only consider mixed Hodge structures over $\mathbb{Q}$, and the cohomology is understood as that with rational coefficients.

Proof. This is a well-known fact to experts. However, we give a complete proof here, due to a lack of good reference. For each $0 \le p \le D$, let $U_p = A_p - A_{p-1}$ and $j_p : U_p \hookrightarrow A_p$ be the open inclusion. Let $i_p : A_{p-1} \hookrightarrow A_p$ be the closed embedding. We then obtain a short exact sequence of sheaves on A_p:

$$0 \to (j_p)_! j_p^{-1}\underline{\mathbb{Q}}_{A_p} \to \underline{\mathbb{Q}}_{A_p} \to (i_p)_* i_p^{-1}\underline{\mathbb{Q}}_{A_p} \to 0 \tag{2.1.4}$$

where $\underline{\mathbb{Q}}_{A_p}$ is the constant sheaf on A_p. Take the hypercohomology with compact support, we obtain a long exact sequence in the abelian category of mixed Hodge structures (Definition/Proposition 2.5):

$$\ldots \to H_c^i(U_p) \xrightarrow{\alpha_p} H_c^i(A_p) \xrightarrow{\beta_p} H_c^i(A_{p-1}) \xrightarrow{\delta_p} H_c^{i+1}(U_p) \to \ldots \tag{2.1.5}$$

We can now construct an *exact couple* [McC01, Section 2.2] from the long exact sequences associated to the triples (U_p, A_p, A_{p-1}) as follows: Take

$$D := \oplus_{p,q} D^{p,q}, D^{p,q} := H_c^{p+q-1}(X_{p-1}); E := \oplus_{p,q} E^{p,q}, E^{p,q} := H_c^{p+q}(U_p).$$

Define morphisms of $\mathbb{Q}$-modules $i : D \to D$, $j : D \to E$, and $k : E \to D$ as follows: Let

$$\begin{aligned} i|_{D^{p+1,q}} &= \beta_p : D^{p+1,q} = H_c^{p+q}(X_p) \to D^{p,q+1} = H_c^{p+q}(X_{p-1});\\ j|_{D^{p,q+1}} &= \delta_p : D^{p,q+1} = H_c^{p+q}(X_{p-1}) \to E^{p,q+1} = H_c^{p+q+1}(U_p);\\ k|_{E^{p,q+1}} &= \alpha_p : E^{p,q+1} = H_c^{p+q+1}(U_p) \to D^{p+1,q+1} = H_c^{p+q+1}(X_p). \end{aligned}$$

It's easy to check that we have obtained an exact couple $C = \{D, E, i, j, k\}$ of bi-graded $\mathbb{Q}$-modules

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

such that the bi-degrees of the morphisms are: $\deg(i) = (-1, 1), \deg(j) = (0, 0)$, and $\deg(k) = (1, 0)$. Recall that, an *exact couple* $C = \{D, E, i, j, k\}$ is a diagram of bi-graded $\mathbb{Q}$-modules as above, with i, j, k $\mathbb{Q}$-module homomorphisms, such that, the diagram is exact at each vertex. Also, given any exact couple $C = \{D, E, i, j, k\}$, the *derived couple* $C' = C^{(1)} = \{D', E', i', j', k'\}$ of C is defined as follows: Take

$$D' = i(D) = \ker(j), E' = H(E, d) = \mathrm{Ker}(j \circ k)/\mathrm{Im}(j \circ k), \text{where } d = j \circ k.$$

and define

$$\begin{aligned} &i' = i|_{i(D)} : D' \to D';\\ &j' : D' \to E' \text{ by } j'(i(x)) = j(x) + dE \in E', \forall x \in D;\\ &k' : E' \to D' \text{ by } k'(e + dE) = k(e), \forall e \in \mathrm{Ker}(d). \end{aligned}$$

Notice that C' is again an exact couple [McC01, Prop.2.7].

In our case, for each $n \ge 0$, let $C^{(n)} = \{D^{(n)}, E^{(n)}, i^{(n)}, j^{(n)}, k^{(n)}\} = (C^{(n-1)})'$ be the n-th derived couple of C. Then, by [McC01, Thm.2.8], the exact couple C induces a spectral sequence $\{E_r, d_r\}, r = 1, 2, \ldots$, where $E_r = E^{(r-1)}$, and $d_r = j^{(r)} \circ k^{(r)}$ has bi-degree $(r, 1 - r)$. In particular, $E_1 = E = E^{*,*}, d_1 = j \circ k$.

To finish the proof of the lemma, we also need to determine $E_\infty = E_r$ for $r >> 0$. By [McC01, Prop.2.9], have $E_r^{p,q} = Z_r^{p,q}/B_r^{p,q}$, where $Z_r^{p,q} = k^{-1}(\mathrm{Im}(i^{r-1}) : D^{p+r,q-r+1} \to D^{p+1,q})$, $B_r^{p,q} = j(\mathrm{Ker}(i^{r-1}) : D^{p,q} \to D^{p-r+1,q+r-1})$. Moreover, $E_\infty^{p,q} = \cap_r Z_r^{p,q}/\cup_r B_r^{p,q}$.

In our case, clearly have $E_\infty^{p,q} = E_r^{p,q}$ for $r >> 0$. Moreover, for $r >> 0$, we see that $i^{r-1} = 0 : D^{p,q} = H_c^{p+q-1}(A_{p-1}) \to D^{p-r+1,q+r-1} = H_c^{p+q-1}(A_{p-r} = \emptyset) = 0$, and $j = \delta_p : D^{p,q} = H_c^{p+q-1}(A_{p-1}) \to E^{p,q} = H_c^{p+q}(U_p)$, so $B_r^{p,q} = \mathrm{Im}(\delta_p : H_c^{p+q-1}(A_{p-1}) \to H_c^{p+q}(U_p)) = \mathrm{Ker}(\alpha_p : H_c^{p+q}(U_p) \to H_c^{p+q}(A_p))$. On the other hand, for $r >> 0$, $i^{r-1} = I_p^* : D^{p+r,q-r+1} = H_c^{p+q}(A_{p+r-1} = A_D) \to D^{p+1,q} = H_c^{p+q}(A_p)$ is the natural morphism induced by the inclusion $I_p : A_p \hookrightarrow A_D$, and $k = \alpha_p : E^{p,q} = H_c^{p+q}(U_p) \to D^{p+1,q} = H_c^{p+q}(A_p)$. So, $Z_r^{p,q} = \alpha_p^{-1}(I_p^* : H_c^{p+q}(A_D) \to H_c^{p+q}(A_p))$. Therefore, we have $E_r^{p,q} = \alpha_p^{-1}(\mathrm{Im}(I_p^*))/\mathrm{Ker}(\alpha_p) \cong \mathrm{Im}(I_p^*) \cap \mathrm{Im}(\alpha_p) = \mathrm{Im}(I_p^*) \cap \mathrm{Ker}(\beta_p)$, where the last 2 equalities follow from the following commutative diagram with exact rows, in which all the squares are fiber products:

$$\begin{tikzcd}
 & & & \mathrm{Im}(I_p^*) \cap \mathrm{Im}(\alpha_p) \arrow[dr, dotted, hook] \arrow[dd, dotted, hook] & & \\
0 \arrow[r] & \mathrm{Ker}(\alpha_p) \arrow[r] \arrow[dd, "Id"] & \alpha_p^{-1}(\mathrm{Im}(I_p^*)) \arrow[ur, dotted, two heads] \arrow[rr] \arrow[dd, hook] & & \mathrm{Im}(I_p^*) \arrow[dd, hook] & \\
 & & & \mathrm{Im}(\alpha_p) \arrow[dr, dotted, hook] & & \\
0 \arrow[r] & \mathrm{Ker}(\alpha_p) \arrow[r] & H_c^{p+q}(U_p) \arrow[ur, dotted, two heads] \arrow[rr, "\alpha_p"] & & H_c^{p+q}(A_p) \arrow[r, "\beta_p"] & H_c^{p+q}(A_{p-1})
\end{tikzcd}$$

Let $F^p H_c^{p+q}(X_D) := \mathrm{Ker}(I_{p-1}^*)$. Clearly, the identity of inclusions $I_{p-1} = I_p \circ i_p : A_{p-1} \overset{i_p}{\hookrightarrow} A_p \overset{I_p}{\hookrightarrow} A_D$ induces $I_{p-1}^* = i_p^* \circ I_p^* = \beta_p \circ I_p^* : H_c^{p+q}(X_D) \xrightarrow{I_p^*} H_c^{p+q}(A_p) \xrightarrow{\beta_p} H_c^{p+q}(A_{p-1})$. So we obtain a filtration $H_c^{p+q}(X_D) = F^0 \supset F^1 \supset \ldots \supset F^D \supset F^{D+1} = 0$ for $H_c^{p+q}(X_D)$. Thus, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
 & & 0 \arrow[r, dotted] \arrow[d] & E_\infty^{p,q} \arrow[r, dotted, dash] \arrow[d] & \\
0 \arrow[r] & \mathrm{Ker}(I_p^*) \arrow[r] \arrow[d, hook] & H_c^{p+q}(X_D) \arrow[r] \arrow[d, "Id"] & \mathrm{Im}(I_p^*) \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & \mathrm{Ker}(I_{p-1}^*) \arrow[r] \arrow[d] & H_c^{p+q}(X_D) \arrow[r] \arrow[d] & \mathrm{Im}(I_{p-1}^*) \arrow[r] & 0 \\
 \arrow[r, dotted] & F^p/F^{p+1} \arrow[r, dotted] & 0 & &
\end{tikzcd}$$

By the five lemma, we then have the natural isomorphism $E_\infty^{p,q} \cong F^p/F^{p+1}(H_c^{p+q}(X_D))$. Thus, the spectral sequence $\{E_r^{p,q}, d_r\}$ converges to $H_c^{p+q}(X_D)$, with the first page given by $E_1^{p,q} = H_c^{p+q}(U_p) = H_c^{p+q}(A_p \setminus A_{p-1})$. Finally, the compatibility with MHS is automatic, as all the morphisms in the previous construction, hence in the spectral sequence, are morphisms in the abelian category of mixed Hodge structures over $\mathbb{Q}$. $\square$

2.2. **Application.** The spectral sequence in the previous subsection allows us to draw some conclusions about the cohomology of the augmentation variety. We begin with some preliminaries on mixed Hodge structures, mainly due to Deligne [Del71, Del74]. A general reference is [PS08]. We only review the part which is most relevant to us.

Definition/Proposition 2.5. ([Del71, Del74] or [PS08])

(1) Let X be a complex algebraic variety, for each j there exists an increasing weight filtration
$$0 = W_{-1} \subset W_0 \subset \ldots \subset W_j = H_c^j(X) = H_c^j(X, \mathbb{Q})$$
and a decreasing Hodge filtration
$$H_c^j(X)^{\mathbb{C}} = H_c^j(X, \mathbb{C}) = F^0 \supset F^1 \supset \ldots \supset F^m \supset F^{m+1} = 0$$
such that the filtration F induces a pure Hodge structure of weight l on the complexification of the graded pieces $\mathrm{Gr}_l^W = W_l/W_{l-1}$ of the weight filtration: for each $0 \leq p \leq l$, we have
$$\mathrm{Gr}_l^{W^{\mathbb{C}}} = F^p \mathrm{Gr}_l^{W^{\mathbb{C}}} \oplus \overline{F^{l-p+1} \mathrm{Gr}_l^{W^{\mathbb{C}}}}.$$

(2) If X is smooth and projective, then $H_c^j(X) = H^j(X, \mathbb{Q})$ is a pure Hodge structure of weight j, with the Hodge filtration $F^i H^j(X, \mathbb{C}) = \oplus_{p+q=j, p \geq i} H^{p,q}(X)$, induced from the classical Hodge decomposition $H^j(X, \mathbb{C}) = \oplus_{p+q=j} H^{p,q}(X) = H^q(X, \Omega^p)$.
For example, if $X = \mathbb{P}^1(\mathbb{C})$, then $H^2(X) = \mathbb{Q}(-1)$ is the pure Hodge structure of weight 2 on $\mathbb{Q}$, with the Hodge filtration on $H^2(X, \mathbb{C}) = H^{1,1}(X) = \mathbb{C}$ given by $F^1 = \mathbb{C}, F^2 = 0$. Here $\mathbb{Q}(-1)$ is called the (-1)*-th Tate twist* (of the trivial weight 0 pure Hodge structure $\mathbb{Q}$). In general, *define* $\mathbb{Q}(-m) := (\mathbb{Q}(-1))^{\otimes m}$ to be the $(-m)$*-th Tate twist*, that is, a pure Hodge structure of weight $2m$ on $\mathbb{Q}$, with Hodge filtration $F^m = \mathbb{C}, F^{m+1} = 0$.

(3) If we replace $H_c^j(X, \mathbb{Q})$ by any finite dimensional vector spaces V over $\mathbb{Q}$, then (1) gives a mixed $\mathbb{Q}$-Hodge structure (MHS) on V. One standard fact is that, the category of MHSs form an abelian category [PS08, Cor.2.5].

(4) Given any triple (U, X, Z) of complex varieties, with $i : Z \hookrightarrow X$ the closed embedding, and $j : U = X - Z \hookrightarrow X$ the open complement, there exists an induced long exact sequence in the abelian category of MHSs:
$$\ldots \to H_c^*(U) \xrightarrow{j_!} H_c^*(X) \xrightarrow{i^*} H_c^*(Z) \xrightarrow{\delta} H_c^{*+1}(U) \to \ldots$$

Definition 2.6. For any complex algebraic variety X, define the *(compactly supported) mixed Hodge numbers* by
$$h_c^{p,q;j}(X) := \dim_{\mathbb{C}} \mathrm{Gr}_p^F \mathrm{Gr}_{p+q}^W H_c^j(X)^{\mathbb{C}}.$$
Define the *(compactly supported) mixed Hodge polynomial* of X by
$$H_c(X; x, y, t) := \sum_{p,q,j} h_c^{p,q;j}(X) x^p y^q t^j.$$
And, the specialization $E(X; x, y) := H_c(X; x, y, -1)$ is called the *weight polynomial* (or *E-polynomial*) of X.

Definition 2.7. We say, an complex algebraic variety X is *Hodge-Tate type*, if $h_c^{p,q;j} = 0$ whenever $p \neq q$. That is, X is of Hodge-Tate type, if for each j and l, the piece $F^p \cap \overline{F^q}$ of Hodge type (p,q) on the pure Hodge structure $Gr_l^W H_c^j(X)^{\mathbb{C}}$ vanishes whenever $p \neq q$.

Now, we come back to the study of augmentation varieties:

Proposition 2.8. *The MHS on* $H_c^*(\mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{C}))$ *is of Hodge-Tate type.*

Proof. By the previous subsection, the ruling filtration for $A_D = \mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{C})$ induces a spectral sequence $E_1^{p+q} = H_c^{p+q}(A_p \setminus A_{p-1}) \Rightarrow H_c^{p+q}(A_D)$, in the abelian category of mixed Hodge structures over $\mathbb{Q}$. Moreover, $A_p \setminus A_{p-1} = \sqcup_{\rho\in R_p}\mathrm{Aug}_m^{\rho}(T;\mathbb{C})$, where $\mathrm{Aug}_m^{\rho}(T;\mathbb{C}) = \mathrm{Aug}_m^{\rho}(T,\rho_L,\rho_R;\mathbb{C}) \cong (\mathbb{C}^{\times})^{a(\rho)} \times \mathbb{C}^{b(\rho)}$ by Theorem 1.29, with $a(\rho) = -\chi(\rho) + B + n'_L, b(\rho) = r(\rho) + A(\rho_L)$. Hence, $E_1^* = H_c^*(A_p \setminus A_{p-1}) = \oplus_{\rho\in R_p} H_c^*(\mathbb{C}^{\times})^{\otimes a(\rho)} \otimes_{\mathbb{Q}} H_c^*(\mathbb{C})^{\otimes b(\rho)}$, is of Hodge-Tate type (Example 2.12). As each E_{r+1}^* is a subquotient of E_r^*, it follows that E_r^* for all $r \geq 1$, in particular, $E_\infty^* = H_c^*(A_D)$, is also of Hodge-Tate type. □

Also, we have:

Proposition 2.9. $H_c^i(\mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{C})) = 0$ *for* $i < C$*, where* $C = C(T,\rho_L,\rho_R) := (-\chi(\rho) + B + n'_L) + 2(r(\rho) + A(\rho_L))$ *(Theorem 1.29, Remark 1.31) is a constant depending only on* T,ρ_L,ρ_R*. In particular, the cohomology* $H_c^*(\mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{C}))$ *vanishes in the lower-half degrees.*

Proof. In the proof of Proposition 2.8, we observe that $H_c^*((\mathbb{C}^{\times})^{a(\rho)} \times \mathbb{C}^{b(\rho)}) = 0$ if $* < a(\rho) + 2b(\rho) = C$ (Example 2.12). Hence, $E_1^{p,q} = H_c^{p+q}(A_p \setminus A_{p-1}) = 0$ if $p + q < C$. It then follows from the spectral sequence that, E_r^{p+q} for all $r \geq 1$, in particular, $E_\infty^{p+q} = H_c^{p+q}(A_D)$, vanishes if $p + q < C$. □

In the spectral sequence in Lemma 2.4 associated to $A_D = \mathrm{Aug}_m(T,\rho_L,\rho_R;\mathbb{C})$, take the 1st page $E_1^{p,q} = H_c^{p+q}(A_p \backslash A_{p-1})$ and forget the differential d_1, the mixed Hodge structure on $\oplus_p E_1^{p,q} = \oplus_p H_c^{p+q}(A_p \setminus A_{p-1})$ gives the first approximation of the mixed Hodge structure on $H_c^{p+q}(A_D)$. Consider the variety $\widetilde{\mathrm{Aug}}_m(T,\rho_L,\rho_R;k) = \sqcup_\rho \mathrm{Aug}_m^{\rho}(T;k)$ of the disjoint union of the pieces in the ruling decomposition, which is not identical to $\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$ as varieties. We see that $\oplus_p H_c^{p+q}(A_p \setminus A_{p-1}) = H_c^{p+q}(\widetilde{\mathrm{Aug}}_m(T,\rho_L,\rho_R;\mathbb{C}))$. This is a Legendrian isotopy invariant, up to a normalization:

Lemma 2.10. *Given any two Legendrian tangles* $(T,\mu),(T',\mu')$*, and any (generic) Legendrian isotopy* h *between them, there's an isomorphism*

$$\widetilde{\Phi}_h : \widetilde{\mathrm{Aug}}_m(T,\rho_L,\rho_R;k) \times (k^*)^{B(T')} \times k^{\dim'-B(T')} \xrightarrow{\sim} \widetilde{\mathrm{Aug}}_m(T',\rho_L,\rho_R;k) \times (k^*)^{B(T)} \times k^{\dim-B(T)}$$

which induces the natural bijection ϕ_h *in Lemma 1.7 ([Su17, Lem.2.9]) on the underlying sets of* m*-graded normal rulings. Here* $B(T), B(T')$ *is the number of base points on* T, T' *respectively, and* $\dim = \dim \mathrm{Aug}_m(T,\rho_L,\rho_R;k)$ *(resp.* $\dim' = \dim \mathrm{Aug}_m(T',\rho_L,\rho_R;k)$*).*

In particular, the mixed Hodge polynomial of $\widetilde{\mathrm{Aug}}_m(T,\rho_L,\rho_R;\mathbb{C})$*, up to a normalization, is a Legendrian isotopy invariant:*

$$H_c(\mathbb{C}^{\times};x,y,t)^{-B(T)} H_c(\mathbb{C};x,y,t)^{-\dim+B(T)} H_c(\widetilde{\mathrm{Aug}}_m(T,\rho_L,\rho_R;\mathbb{C});x,y,t)$$

$$= \sum_{\rho\in \mathrm{NR}_T^m(\rho_L,\rho_R)} (t+qt^2)^{a(\rho)-B(T)}(qt^2)^{b(\rho)-\dim+B(T)}$$

where $q = xy$.

Proof. By Theorem 1.29, we have $\mathrm{Aug}_m^{\rho}(T;k)\times(k^*)^{B(T')}\times k^{\dim'-B(T')} \cong (k^*)^{a(\rho)+B(T')}\times k^{b(\rho)+\dim'-B(T')}$, and similarly for $\mathrm{Aug}_m^{\phi_h(\rho)}(T';k)\times(k^*)^{B(T)}\times k^{\dim-B(T)}$. By Lemma 1.7, it's already known that $\chi(\phi_h(\rho)) = \chi(\rho)$ for all $\rho\in \mathrm{NR}_T^m(\rho_L,\rho_R)$. Hence, $a(\rho)+B(T') = -\chi(\rho)+B(T)+n_L'+B(T') = a(\phi_h(\rho))+B(T)$ (see Theorem 1.29). Also, by Remark 1.31, $-\chi(\rho)+2r(\rho)$ is independent of ρ. It follows that

$$\begin{aligned}
\dim &= \max_{\rho\in \mathrm{NR}_T^m(\rho_L,\rho_R)}\{-\chi(\rho)+B(T)+n_L'+r(\rho)+A(\rho_L)\}\\
&= \max_{\rho}\{\frac{-\chi(\rho)}{2}\}+\frac{-\chi(\rho)}{2}+r(\rho)+B(T)+n_L'+A(\rho_L)\\
&= \max\{\frac{-\chi(\phi_h(\rho))}{2}\}+\frac{-\chi(\phi_h(\rho))}{2}+r(\rho)+B(T)+n_L'+A(\rho_L)\\
&= \dim' - b(\phi_h(\rho)) - B(T') + b(\rho) + B(T)
\end{aligned}$$

That is, $b(\phi_h(\rho))+\dim-B(T) = b(\rho)+\dim'-B(T')$. Therefore, $\mathrm{Aug}_m^{\rho}(T;k)\times(k^*)^{B(T')}\times k^{\dim'-B(T')} \cong \mathrm{Aug}_m^{\phi_h(\rho)}(T';k)\times(k^*)^{B(T)}\times k^{\dim-B(T)}$. This ensures the existence of an isomorphism $\widetilde{\Phi}_h$. □

Remark 2.11. Notice that, by Theorem 1.29, if we instead work with the augmentation varieties $\mathrm{Aug}_m(T,\epsilon_L,\rho_R;k)$, all the previous discussions in this section still apply, possibly up to a different normalization.

2.3. **Examples.**

Example 2.12. We begin with some preliminary examples.

(1) Take $X = \mathbb{C}^\times$. For example, take T to be the standard Legendrian unknot with 2 base points, with one on the right cusp, then $X = \mathrm{Aug}_m(T;\mathbb{C}) = \mathbb{C}^\times$. Let $Y = \mathbb{P}^1(\mathbb{C})$, and $j: X = \mathbb{C}^\times \hookrightarrow Y$ be the open inclusion, with the closed complement $i: Z = \{0,\infty\} \hookrightarrow Y$. From the classical Hodge theory, we know $H_c^*(Y) = \mathbb{Q}[0]\oplus\mathbb{Q}(-1)[-2]$, where $[\cdot]$ corresponds to the cohomological degree shifting. That is, $H_c^*(Y)$ is the pure Hodge structure $\mathbb{Q}$ in cohomology degree 0, $\mathbb{Q}(-1)$ in cohomology degree 2, and 0 otherwise. Similarly, $H_c^*(Z) = \mathbb{Q}^2[0]$. Now, by Definition/Proposition 2.5, the triple (X, Y, Z) induces a long exact sequence of mixed Hodge structures:

$$\begin{aligned}
&0\to H_c^0(X)\to H_c^0(Y)=\mathbb{Q}\to H_c^0(Z)=\mathbb{Q}^2\\
&\to H_c^1(X)\to H_c^1(Y)=0\to H_c^1(Z)=0\\
&\to H_c^2(X)\to H_c^2(Y)=\mathbb{Q}(-1)\to H_c^2(Z)=0
\end{aligned}$$

Together with the knowledge about the cohomology of X, it implies that $H_c^*(X) = \mathbb{Q}[-1]\oplus\mathbb{Q}(-1)[-2]$ as MHSs. Thus, $H_c(X;x,y,t) = t + xyt^2$.

(2) Similarly, take $X = \mathbb{C}$. We see that $H_c^*(X) = \mathbb{Q}(-1)[-2]$. Thus, $H_c(X;x,y,t) = xyt^2$.

(3) Now, take $X = (\mathbb{C}^\times)^a\times\mathbb{C}^b$. The Künneth formula implies that $H_c^*(X) = H_c^*(\mathbb{C}^\times)^{\otimes a}\otimes_{\mathbb{Q}} H_c^*(\mathbb{C})^{\otimes b} = (\mathbb{Q}[-1]\oplus\mathbb{Q}(-1)[-2])^{\otimes a}\otimes_{\mathbb{Q}}(\mathbb{Q}(-1)[-2])^{\otimes b}$. Thus, $H_c(X;x,y,t) = (t+xyt^2)^a(xyt^2)^b$. In particular, X is of Hodge-Tate type, and $H_c^*(X)$ vanishes if $* < a+2b$.

Example 2.13. Take (Λ, μ) be the Legendrian right-handed trefoil knot as in Figure 2.1, with $B(\Lambda) = 2$ base points placed on the 2 right cusps. Clearly, the rotation number $r = 0$. As in the figure, denote the generic vertical lines by $x = x_i, 0 \leq i \leq 3$. Take the Legendrian tangle $(T, \mu) := (\Lambda, \mu)|_{\{x_0 < x < x_3\}}$, this is the example in [Su17, Example.4.25]. So, $T = E_1 \circ E_2 \circ E_3$ is a composition of 3 elementary Legendrian tangles, where $E_i = \Lambda|_{\{x_{i-1} < x < x_i\}}$ for $1 \leq i \leq 3$. Denote $T_i = \Lambda|_{\{x_0 < x < x_i\}} = E_1 \circ \ldots \circ E_i$ for $1 \leq i \leq 3$. As usual, for each i, label the strands of T over $x = x_i$ from top to bottom by $1, 2, \ldots, s_i$. For simplicity, take $m \neq 1$. Recall [Su17, Example.2.13] that $\mathrm{NR}^m_{T_L} = \{(\rho_L)_1 = (12)(34), (\rho_L)_2 = (13)(24)\}$, $\mathrm{NR}^m_{T_R} = \{(\rho_R)_1 = (12)(34), (\rho_R)_2 = (13)(24)\}$.

Use the notations in [Su17, Example.4.25], recall that

(1) $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_1; k) \cong \{(x_i)_{i=1}^3 \in k^3 | x_1 + x_3 + x_1x_2x_3 \neq 0\} = k^* \times k \sqcup k^* \times k \sqcup (k^*)^3$.
(2) $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_2; k) \cong \{(x_i)_{i=1}^3 \in k^3 | x_1 + x_3 + x_1x_2x_3 = 0\} \cong \{(x_i)_{i=1}^2 \in k^2 | 1 + x_1x_2 \neq 0\} = k \sqcup (k^*)^2$.
(3) $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_1; k) \cong \{(x_i)_{i=1}^3 \in k^3 | 1 + x_2x_3 \neq 0\} = k^2 \sqcup k \times (k^*)^2$.
(4) $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_2}, (\rho_R)_2; k) = \{(x_i)_{i=1}^3 | 1 + x_2x_3 = 0\} \cong \{(x_i)_{i=1}^2 \in k^2 | x_2 \neq 0\} = k \times k^*$.

where in each case above, the last equality corresponds to the ruling decomposition. Also, (x_1, x_2, x_3) corresponds to the augmentation $\epsilon \in \mathrm{Aug}_m(T; k)$ defined by $\epsilon(a_i) = x_i$, and $\epsilon|_{T_L} = \epsilon_{(\rho_L)_1}$ (resp. $\epsilon_{(\rho_L)_2}$) in (1), (2) (resp. (3), (4)).

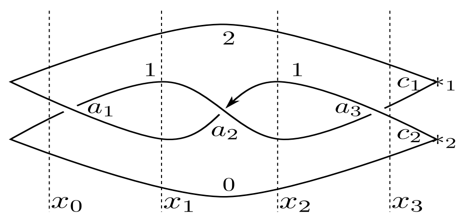

FIGURE 2.1. (Λ, μ) = the Legendrian right-handed trefoil knot with 2 base points $*_1, *_2$ at the right cusps c_1, c_2 respectively. a_1, a_2, a_3 are the crossings, and the numbers encode the Maslov potential values on each strand. Moreover, define Legendrian tangles $T_i := \Lambda|_{\{x_0 < x < x_i\}}$, $1 \leq i \leq 3$, and $T = T_3$.

We want to compute the mixed Hodge polynomial for each case. *Define* for each Legendrian tangle T_i, the augmentation variety $\mathrm{Aug}_m(T_i, \epsilon_L; k) := \{\epsilon \in \mathrm{Aug}_m(T_i; k) : \epsilon|_{(T_i)_L} = \epsilon_L\}$. *Denote* $k := \mathbb{C}$. For each i, *denote* the pairs of endpoints of $T|_{\{x = x_i\}}$ by a^i_{pq}, $1 \leq p < q \leq 4$. In the computation, we will use the following fact frequently: If $Y = \mathbb{A}^n(k)$ is an affine space, $j : U \hookrightarrow Y$ is an nonempty Zariski open subset, and $i : Z = Y - U \hookrightarrow Y$ is the closed complement. Then $H^*_c(Y) = \mathbb{Q}(-n)[-2n]$ as MHSs, and $\dim_k Z \leq n - 1 \Rightarrow H^{2n-1}_c(Z) = 0 = H^{2n}_c(Z)$. Thus, by Definition/Proposition 2.5, the triple (U, Y, Z) induces exact sequences of MHSs:

$$0 = H^i_c(Y) \to H^i_c(Z) \xrightarrow{\sim} H^{i+1}_c(U) \to H^{i+1}_c(Y) = 0, i + 1 < 2n; \tag{2.3.1}$$
$$0 = H^{2n-1}_c(Z) \to H^{2n}_c(U) \xrightarrow{\sim} H^{2n}_c(Y) = \mathbb{Q}(-n) \to H^{2n}_c(Z) = 0.$$

(a). Firstly, consider the case (2). Note: $\mathrm{Aug}_m(T, \epsilon_{(\rho_L)_1}, (\rho_R)_2; k) \cong \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}, (12)(34); k)$, where (12)(34) is the m-graded normal ruling of $(T_2)_R$. Clearly, $\mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}; k) = \{(x_i)_{i=1}^2 \in k^2\} \cong k^2$. Notice that $1 + x_1x_2 = \epsilon(a^2_{12})$ for any $\epsilon \in \mathrm{Aug}_m(T_2, \epsilon_{(\rho_L)_1}; k)$, it follows that j :

$\mathrm{Aug}_m(T_2,\epsilon_{(\rho_L)_1},(12)(34);k)=\{1+x_1x_2\neq 0\}\hookrightarrow \mathrm{Aug}_m(T_2,\epsilon_{(\rho_L)_1};k)$ is the open embedding, and $i:\mathrm{Aug}_m(T_2,\epsilon_{(\rho_L)_1},(13)(24);k)=\{1+x_1x_2=0\}\cong k^\times\hookrightarrow \mathrm{Aug}_m(T_2,\epsilon_{(\rho_L)_1};k)$ is the closed complement. Hence, (2.3.1) implies that

$$\begin{aligned}&H_c^*(\mathrm{Aug}_m(T_2,\epsilon_{(\rho_L)_1},(12)(34);k))\cong H_c^{*-1}(k^\times)=(\mathbb{Q}[-1]\oplus\mathbb{Q}(-1)[-2])^{*-1},*<4;\\&H_c^4(\mathrm{Aug}_m(T_2,\epsilon_{(\rho_L)_1},(12)(34);k))\cong H_c^4(k^2)=\mathbb{Q}(-2).\end{aligned}$$

Thus,

$$H_c^*(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_2;k))=\mathbb{Q}[-2]\oplus\mathbb{Q}(-1)[-3]\oplus\mathbb{Q}(-2)[-4].$$

(b). Consider the case (1). Note: $\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1};k)=\{(x_i)_{i=1}^3\in k^3\}\cong k^3$, and $x_1+x_3+x_1x_2x_3=\epsilon(a_{12}^3)$ for any $\epsilon\in\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1};k)$. So $j:\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_1;k)=\{x_1+x_3+x_1x_2x_3\neq 0\}\hookrightarrow \mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1};k)$ is the open embedding, and $i:\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_2;k)=\{x_1+x_3+x_1x_2x_3=0\}\hookrightarrow\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1};k)$ is the closed complement. Hence, (2.3.1) implies that

$$\begin{aligned}&H_c^*(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_1;k))\cong H_c^{*-1}(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_2;k))\\&=(\mathbb{Q}[-2]\oplus\mathbb{Q}(-1)[-3]\oplus\mathbb{Q}(-2)[-4])^{*-1},*<6;\\&H_c^6(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_1;k))\cong H_c^6(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1};k))=\mathbb{Q}(-3).\end{aligned}$$

That is,

$$H_c^*(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_1;k))=\mathbb{Q}[-3]\oplus\mathbb{Q}(-1)[-4]\oplus\mathbb{Q}(-2)[-5]\oplus\mathbb{Q}(-3)[-6].$$

(c). Consider the case (4). As $\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_2;k)\cong k^\times\times k$, by Example 2.12, we immediately have:

$$H_c^*(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_2;k))=\mathbb{Q}(-1)[-3]\oplus\mathbb{Q}(-2)[-4].$$

(d). Finally, consider the case (3). Note: $\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2};k)=\{(x_i)_{i=1}^3\in k^3\}\cong k^3$, and $1+x_2x_3=\epsilon(a_{12}^3)$ for any $\epsilon\in\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2};k)$. So $j:\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_1;k)\cong\{(x_i)_{i=1}^3\in k^3|1+x_2x_3\neq 0\}\hookrightarrow\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2};k)$ is the open embedding, and $i:\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_2;k)=\{1+x_2x_3=0\}\cong k^\times\times k\hookrightarrow\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2};k)$ is the closed complement. Hence, (2.3.1) implies that

$$\begin{aligned}&H_c^*(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_1;k))\cong H_c^{*-1}(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_2;k))\\&=(\mathbb{Q}(-1)[-3]\oplus\mathbb{Q}(-2)[-4])^{*-1},*<6;\\&H_c^6(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_1;k))\cong H_c^6(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2};k))=\mathbb{Q}(-3).\end{aligned}$$

That is,

$$H_c^*(\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_2},(\rho_R)_1;k))=\mathbb{Q}(-1)[-4]\oplus\mathbb{Q}(-2)[-5]\oplus\mathbb{Q}(-3)[-6].$$

Note: $\mathrm{Aug}_m(\Lambda;k)\cong\mathrm{Aug}_m(T,\epsilon_{(\rho_L)_1},(\rho_R)_1;k)$, so we also have $H_c^*(\mathrm{Aug}_m(\Lambda;k))=\mathbb{Q}[-3]\oplus\mathbb{Q}(-1)[-4]\oplus\mathbb{Q}(-2)[-5]\oplus\mathbb{Q}(-3)[-6]$. In particular, the mixed Hodge polynomial is given by $H_c(\mathrm{Aug}_m(\Lambda;k);x,y,t)=t^3+qt^4+q^2t^5+q^3t^6$, where $q=xy$. Clearly, $B(\Lambda)=2$, and $\dim=\dim\mathrm{Aug}_m(\Lambda;k)=3$. It follows that,

$$P_\Lambda^m(q,t)=(t+qt^2)^{-B(\Lambda)}(qt^2)^{-\dim+B(\Lambda)}H_c(\mathrm{Aug}_m(\Lambda;k);x,y,t)=\frac{1+q^2t^2}{(1+qt)qt}$$

gives the 2-variable invariant generalizing the ruling polynomial (Corollary 3.11).

3. 'Invariance' of augmentation varieties

In this section, we study the compatible properties of the augmentation varieties for Legendrian tangles under a Legendrian isotopy. In the case of Legendrian knots Λ, the 'invariance' of augmentation varieties follows immediately from the invariance of the stable tame isomorphism class of the LCH DGA $\mathcal{A}(\Lambda)$ [ENS02,HR15]. This approach may be generalized directly to show the 'invariance' of the *full augmentation varieties* associated to Lengendrian tangles. However, we also want the 'invariance' of *augmentation varieties with fixed boundary conditions*, when the situation is more complicated. Here, we will pursue a different approach, i.e. a tangle approach as in [Su17], through which we can reduce the problem to a local one, when the Legendrian tangles in question are simple enough, for example as in Figure 1.2.

3.1. **The identification between augmentations and A-form MCSs.**

We firstly present with some details the identification between augmentations and A-form MCSs for Legendrian tangles, sketched in [Su17, Section.5.1]. This is simply a direct generalization of the case for oriented Legendrian knots in nearly plat positions, given in [HR15, Thm.5.2].

As usual, fix an open interval $U \subset \mathbb{R}_x$ and let T be a Legendrian tangle front in $U \times \mathbb{R}_z$, equipped with a $\mathbb{Z}/2r$-valued Maslov potential μ, base points $*_1, \ldots, *_B$ so that each connected component containing a right cusp has at least one base point. Let n_L and n_R be the number of left and right end-points of T respectively. Fix a base field k and an nonnegative integer m dividing $2r$.

3.1.1. *Morse complex sequences.* We start by reviewing some basic concepts.

Definition 3.1. A *handleslide H_r* of T is a *coefficient $r \in k$*, together with a *vertical line segment* in $U \times \mathbb{R}_z$ avoiding the crossings and cusps, whose end-points lying on two strands of T. A *labeled base point c_r* of T is a base point on T away from the crossings and cusps, together with a coefficient $r \in k^*$. A handleslide H_r of T is called *m-graded* if $r = 0$ or its end-points belong to 2 strands having the same Maslov potential value modulo m. An *elementary tangle* of T is the subset (tangle) of T within some vertical strip containing a single crossing, a single cusp, a single handleslide, or a single labeled base point.

Definition 3.2. A *(m-graded) Morse Complex Sequence (MCS)*[5] over k of T is a triple $C = (\{(C_l, d_l)\}, \{x_l\}, H)$ such that:

1. H is a collection of m-graded handleslides with coefficients in k and labeled base points with coefficients in k^*;
2. $\{x_l\}$ is an increasing sequence of x-coordinates $x_0 < x_1 < \ldots < x_M$, such that $\overline{U} = [x_0, x_M]$ and the vertical lines $x = x_l$ decompose the tangle with handleslides $T \cup H$ into elementary tangles;
3. For each l, (C_l, d_l) is a $\mathbb{Z}/m$-graded complex over k such that: C_l is the free k-module generated by $e_1, \ldots, e_{s_l}$, corresponding to the points of $T \cap \{x = x_l\}$ labeled from top to bottom, with the grading $|e_i| = \mu(i)(\mathrm{mod} m)$; The differential d_l has degree -1 and is lower-triangular, i.e. $< d_l e_i, e_j >= 0$ if $i \geq j$, where $< d_l e_i, e_j >$ denotes the coefficient of e_j in $d_l e_i$ relative to the basis $e_1, \ldots, e_{s_l}$;

[5]Note: The MCSs here are known as 'm-graded Morse complex sequences with simple left cusps' in [HR15]. Also, the usage of different sign conventions leads to a slightly different definition of MCS.

(4) For each l, if the strands k and $k+1$ at $x = x_l$ meet at a crossing (resp. left cusp) near (= rightly before or after) $x = x_l$, then $< d_l e_k, e_{k+1} >= 0$ (resp. $< d_l e_k, e_{k+1} >= (-1)^{\mu(k)}$). If they meet at an unmarked right cusp near $x = x_l$, then $< d_l e_k, e_{k+1} >= -(-1)^{\mu(k)}$. If they meet at the marked right cusp with base-point $*_i$ near $x = x_l$, then $< d_l e_k, e_{k+1} >= -(-1)^{\mu(k)} s_i$, for some invertible element $s_i \in k^\times$. In this case, we say that C assigns the value s_i to the base point $*_i$;

(5) For each $0 \leq l < M$, the complexes (C_l, d_l) and (C_{l+1}, d_{l+1}) are related by the following conditions, depending on the elementary tangle T_l of T between x_l and x_{l+1}:

(a) If T_l contains a *crossing* between strands k and $k+1$ (labeled from top to bottom), then there's an (not necessarily filtered) isomorphism of $\mathbb{Z}/m$-graded complexes $\varphi : (C_l, d_l) \to (C_{l+1}, d_{l+1})$ via[6]

$$\varphi(e_i) = \begin{cases} e_i & i \neq k, k+1 \\ e_{k+1} & i = k \\ e_k & i = k+1 \end{cases}$$

(b) If T_l contains a *handleslide* with coefficient r between strands j and k, $j < k$, then there's an isomorphism of $\mathbb{Z}/m$-graded filtered complexes $\varphi : (C_l, d_l) \to (C_{l+1}, d_{l+1})$ via

$$\varphi(e_i) = \begin{cases} e_i & i \neq j \\ e_j - re_k & i = j \end{cases}$$

(c) If T_l contains a *right cusp* between strands k and $k+1$, then there's a surjective morphism of $\mathbb{Z}/m$-graded filtered complexes $\varphi : (C_l, d_l) \to (C_{l+1}, d_{l+1})$ with kernel $\mathrm{Span}\{e_k, d_l e_k\}$, defined by

$$\varphi(e_i) = \begin{cases} e_i & i < k \\ e_{i-2} & i > k+1 \end{cases}$$

and $\varphi(e_k) = 0, \varphi(d_l e_k) = 0$. Notice that the quotient $(C_l, d_l)/\mathrm{Span}\{e_k, d_l e_k\}$ is freely generated by $[e_i], i \neq k, k+1$ as a k-module, according to the defining condition (4).

(d) If T_l contains a *left cusp* between strands k and $k+1$, then (C_{l+1}, d_{l+1}) is a direct sum of (C_l, d_l) and the acyclic complex $(\mathrm{Span}\{e_k, e_{k+1}\}, d_{l+1} e_k = te_{k+1})$ for some $t \in k^\times$ as a $\mathbb{Z}/m$-graded filtered complex, via the morphism $\varphi : (C_l, d_l) \hookrightarrow (C_{l+1}, d_{l+1})$:

$$\varphi(e_i) = \begin{cases} e_i & i < k \\ e_{i+2} & i \geq k \end{cases}$$

(e) If T_l contains a *labeled base point* c_r with coefficient $r \in k^*$ on the strand k, then there's an isomorphism of $\mathbb{Z}/m$-graded filtered complexes $\varphi : (C_l, d_l) \to (C_{l+1}, d_{l+1})$ via

$$\varphi(e_i) = \begin{cases} e_i & i \neq k \\ re_k & i = k \end{cases}$$

Remark 3.3. By definition, m-graded MCSs satisfy a *sheaf property* similar to that of augmentation varieties for Legendrian tangles.

[6]Note: Here we have used the same notations $\{e_i\}$ for 2 different bases, one for each of C_l, C_{l+1}. We will use this convention throughout the context.

Similarly as in [HR15], we have

Lemma 3.4 ([HR15, Prop.4.2]). *A MCS is uniquely determined by its handleslide set H and initial complex (C_0, d_0).*

Conversely,

Lemma 3.5 ([HR15, Prop.4.3]). *Given an initial complex (C_0, d_0) for T at $x = x_0$, satisfying the conditions (3), (4) in Definition 3.2, a m-graded handleslide set H is the handleslide set of a m-graded MCS with initial complex (C_0, d_0) if and only if, when inductively define the complexes (C_l, d_l) from left to right, have $< d_l e_k, e_{k+1} >= 0, -(-1)^{\mu(k)}$ or $-(-1)^{\mu(k)} s$ for some $s \in k^\times$ whenever $x = x_l$ precedes a crossing, a unmarked right cusp or a marked right cusp between strands k and $k+1$ respectively.*

3.1.2. *A-form MCSs.*

Definition 3.6. Given a m-graded MCS C on T, C is called an *A-form MCS*[7] if its handleslide set is arranged as follows:

(1) There's a handleslide with coefficient in k immediately to the left of a m-graded crossing. The handleslide connects the 2 crossing strands.
(2) There's a labeled base point with coefficient in k^* at each base point (excluding the marked right cusps) of T.
(3) If $m = 1$, there's a handleslide immediately to the left of a right cusp. The handleslide connects the 2 strands meeting at the cusp.

Denote by $MCS_m^A(T; k)$ by the set of all m-graded A-form MCSs over k for T, again the *sheaf property* is satisfied.

Definition 3.7. Given any two m-graded A-form MCSs $C = (\{(C_l, d_l)\}, \{x_l\}, H)$, and $C' = (\{(C_l, d'_l)\}, \{x_l\}, H')$ on T, an *isomorphism* between C, C' is a collection of isomorphisms $\phi = \{\phi_l\}$, where $\phi_l : (C_l, d_l) \xrightarrow{\sim} (C_l, d'_l)$ is an isormorphism of m-graded filtered complexes, such that they commute with the maps φ's in Definition 3.2.

Let T_L be the left piece of T, i.e. T_L consists of n_L parallel strands, equipped with the induced Maslov potential μ_L. By Definition/Proposition 1.13, we have an inclusion of DGAs $\mathcal{A}(T_L) \hookrightarrow \mathcal{A}(T)$ with $\mathcal{A}(T_L) = \mathbb{Z} < a_{ij}, 1 \leq i < j \leq n_L >$, where a_{ij} corresponds to the pair of left end-points i, j of T.

Theorem 3.8. *For any Legendrian tangle front T, there's a natural isomorphism*

$$\Theta : \mathrm{Aug}_m(T; k) \xrightarrow{\sim} MCS_m^A(T; k)$$

which commutes with restrictions. The map Θ is defined as follows:
Let ϵ be a m-graded augmentation of T over k, by Lemma 3.4 it suffices to associate to ϵ a handleslide set H and an initial complex (C_0, d_0) :

(1) (C_0, d_0) : $C_0 = \mathrm{Span}\{e_i, 1 \leq i \leq n_L\}$ where e_i corresponds to the left end-point i of T, with grading $|e_i| = \mu(i)(\mathrm{mod} m)$; $< d_0 e_i, e_j >= (-1)^{\mu_L(i)} \epsilon(a_{ij})$ for $i < j$, and 0 otherwise;

[7] 'A' stands for 'Augmentation'.

(2) H : *For each m-graded crossing q of T, take a handleslide immediately to the left of q as in Definition 3.6, with coefficient* $-\epsilon(q)$*; If* $m = 1$*, for each right cusp q, also add a handleslide immediately to the left of q, with coefficient* $-\epsilon(q)$*;*
For each base point $*$ *(excluding the marked right cusps) with corresponding generator t in the DGA associated to T, take a labeled base point with coefficient* $\epsilon(t)$ *(resp.* $\epsilon(t)^{-1}$*) if the orientation of the strand containing* $*$ *is right-moving (resp. left-moving).*

Moreover, if q is a right cusp marked with the base point $*_i$*, under the identification the value (see Definition 3.2) assigned to the base point is* $\epsilon(t_i)^{\sigma(q)}$*.*

Proof of theorem 3.8. Cut the Legendrian tangle T into elementary Legendrian tangles: a single crossing, a single left cusp, a single right cusp, or a single base point excluding the marked right cusps. Recall that the augmentation variety $\mathrm{Aug}_m(T;k)$ (resp. the set of A-form MCSs $MCS_m^A(T;k)$) satisfies the sheaf property, so can be written as a fiber product of augmentation varieties (resp. the sets of A-form MSCs) for elementary Legendrian tangles over those for trivial Legendrian tangles. Hence, it suffices to show the theorem for the following simple Legendrian tangles: n parallel strands, a single crossing, a single left cusp, a single right cusp, and a single base point (excluding the marked right cusps). In these cases, the theorem reduces to Example 1.20 and Lemma 5.2 in [Su17], whose proof can be done by a direct calculation. □

From now on, we will always use the identification between augmentations and A-form MCSs.

3.1.3. *Handleslide moves.* Given a trivial Legendrian tangle (T, μ) of n paralle strands, a $\mathbb{Z}/m$-graded handleslide H_r with coefficient r between strands $j < k$, is also equivalent to an $\mathbb{Z}/m$-graded filtered *elementary transformation* $H_r : C(T_L) \xrightarrow{\sim} C(T_R)$ (Definition 1.21):

$$H_r(e_i) = \begin{cases} e_i & i \neq j \\ e_j - re_k & i = j \end{cases}$$

Similarly, a labeled base point c_r with coefficient $r \in k^*$ on the strand k, is equivalent to an $\mathbb{Z}/m$-graded filtered *elementary transformation* $c_r : C(T_L) \xrightarrow{\sim} C(T_R)$:

$$c_r(e_i) = \begin{cases} e_i & i \neq k \\ re_k & i = k \end{cases}$$

Definition 3.9. Let's also *define* an $\mathbb{Z}/m$-graded *unfiltered elementary transformation* $H_r^{\uparrow}$: $C(T_L) \xrightarrow{\sim} C(T_R)$ for $j < k$:

$$H_r^{\uparrow}(e_i) = \begin{cases} e_i & i \neq k \\ e_k - re_j & i = k \end{cases}$$

We can represent $H_r^{\uparrow}$ by an upper arrow between strands j, k with coefficient r, termed as *unfiltered handleslide*. We will use this notion in the proof of Theorem 3.10. For example, see Figure 3.3 (middle) and Figure 3.5 (middle and right).

Also, a single crossing s_k between strands $k, k+1$ in a MCS is equivalent to a $\mathbb{Z}/m$-graded (not necessarily filtered) *elementary transformation* $s_k : C(T_L) \xrightarrow{\sim} C(T_R)$, with T the elementary tangle containing the crossing:

$$s_k(e_i) = \begin{cases} e_i & i \neq k, k+1 \\ e_{k+1} & i = k \\ e_k & i = k+1 \end{cases}$$

There're some identities involving the elementary transformations (represented by handleslides H_r or crossings s_k as above) between $\mathbb{Z}/m$-graded complexes. They can be represented by the *local moves* (or *handleslide moves*) of diagrams as in Figure 3.1: Each diagram represents a composition of elementary transformations with the maps going from left to right, and each local move represents an identity between 2 different compositions.

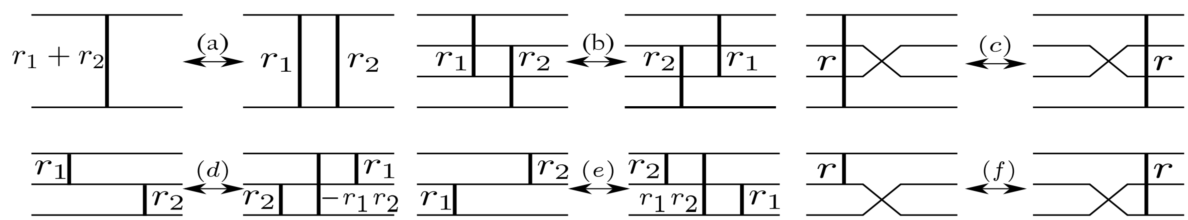

FIGURE 3.1. Local moves of handleslides in a Legendrian tangle T = identities between different compositions of elementary transformations. The moves shown do not illustrate all the possibilities.

More precisely, the possible local moves in a Legendrian tangle (T, μ) are as follows (see also [HR15, Section.6]):

Type 0: (Introduce or remove a trivial handleslide) Introduce or remove a handleslide with coefficient 0 and endpoints on two strands with the same Maslov potential value modulo m.

Type 1: (Slide a handleslide past a crossing) Suppose T contains one single crossing between strands k and $k+1$, and exactly one handleslide h between strands $i < j$, with $(i, j) \neq (k, k+1)$. We may slide h (either left or right) past the crossing such that the endpoints of h remain on the same strands of T. See Figure 3.1 (c),(f) for two such examples.

Type 2: (Interchange the positions of two handleslides) If T contains exactly two handleslides h_1, h_2 between strands $i_1 < j_1$, and $i_2 < j_2$, with coefficients r_1, r_2 respectively. If $j_1 \neq i_2$ and $i_1 \neq j_2$, we may interchange the positions of the handleslides, see Figure 3.1 (b) for an example; If $j_1 = i_2$ (resp. $i_1 = j_2$) and h_1 is to the left of h_2, we may interchange the positions of h_1, h_2, and introduce a new handleslide between strands i_1, j_2 (resp. i_2, j_1), with coefficient $-r_1 r_2$ (resp. $r_1 r_2$), see Figure (d) (resp. (e)).

Type 3: (Merge two handleslides) Suppose T contains exactly two handleslides h_1, h_2 between the same two strands, with coefficients r_1, r_2, respectively. We may merge the two handleslides into one between the same two strands, with coefficient $r_1 + r_2$, see Figure 3.1 (a).

Type 4: (Introduce two canceling handleslides) Suppose T contains no crossings, cusps or handleslides. We may introduce two new handleslides between the same two strands, with coefficients $r, -r$, where $r \in k$.

3.2. **Invariance of augmentation varieties up to an affine factor.** Let $\mathrm{Aug}_m^a(T;k)$ be the subvariety of acyclic augmentations of the full augmentation variety $\mathrm{Aug}_m(T;k)$. That is, $\mathrm{Aug}_m^a(T;k) = \sqcup_{\rho_L,\rho_R}\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$ where ρ_L,ρ_R run over all m-graded normal rulings of T_L,T_R respectively.

Theorem 3.10. *Given any 2 Legendrian tangles $(T,\mu),(T',\mu')$ so that each right cusp of T,T' is marked, and h is a (generic) Legendrian isotopy between them, there's an isomorphism*

$$\Phi_h : \mathrm{Aug}_m^a(T;k)\times(k^*)^{\alpha}\times k^{\beta} \xrightarrow{\sim} \mathrm{Aug}_m^a(T';k)\times(k^*)^{\alpha'}\times k^{\beta'}$$

for some nonnegative integers $\alpha,\alpha',\beta,\beta'$. Moreover, under the obvious restriction maps, the isomorphism Φ_h commutes with the identity map $Id : \mathrm{Aug}_m^a(T_L;k) \xrightarrow{\sim} \mathrm{Aug}_m^a(T'_L;k)$, and is compatible with the ruling decomposition *over $T_R = T'_R$, that is, the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Aug}_m^a(T;k)\times(k^*)^{\alpha}\times k^{\beta} \arrow[r, "\Phi_h"] \arrow[d, "R_T(\cdot)|_{T_R}"'] & \mathrm{Aug}_m^a(T';k)\times(k^*)^{\alpha'}\times k^{\beta'} \arrow[d, "R_{T'}(\cdot)|_{T'_R}"] \\
\mathrm{NR}^m_{T_R} \arrow[r, "Id"] & \mathrm{NR}^m_{T'_R}
\end{tikzcd}$$

Proof of Theorem 3.10. Any Legendrian isotopy is a composition of a finite sequence of simple Legendrian isotopies, such as a smooth isotopy which switches the x-coordinates of two neighboring crossings, or one of the tree types of Legendrian Reidemeister moves. Hence, It suffices to show the case when h is a simple Legendrian isotopy between T,T'.

By cutting the Legendrian tangles T,T' into simple pieces, we can assume $T = X\circ Y\circ Z$ and $T' = X\circ Y'\circ Z$ are compositions of simpler Legendrian tangles, such that Y,Y' are the "minimal" pieces involved in the Legendrian isotopy h.

Let's firstly prove the theorem for Y,Y'. The nontrivial cases are as follows, the other cases are either trivial or similar.

If h is a smooth isotopy between Y,Y', which switches the x-coordinates of two neighboring crossings a,b, as in Figure 3.2. We may assume Y (resp. Y') is the Legendrian tangle shown as in Figure 3.2 (left) (resp. (right)) without the handleslides. Assume a (resp. b) is the crossing between strands $i,i+1$ (resp. $j,j+1$), so $i+1<j$. Denote by s_a, s_b the elementary transformations represented by the crossings a,b respectively, and by H_r (resp. H_s) the handleslide with coefficient $r\in k$ (resp. $s\in k$) to the immediate left of a (resp. b) between the crossing strands of a (resp. b). Denote $C_0 = C_L = C(Y_L) = C(Y'_L), C_R = C(Y_R) = C(Y'_R)$ (Definition 1.21).

Use the identification between augmentations and A-form MCSs, we have isomorphisms

$$\begin{aligned}
&\mathrm{Aug}_m^a(Y;k) \cong \{(d_0,r,s)|(C_i,d_i) \text{ is a } m\text{-graded filtered acyclic complex,}\\
&\quad\text{the handleslides } H_r,H_s \text{ are } m\text{-graded.}\}\\
\cong\ &\{(d_0,r,s)|d_0\in \mathrm{Aug}_m^a(Y_L;k), <d_0e_i,e_{i+1}>=0=<d_0e_j,e_{j+1}>, H_r,H_s\\
&\quad\text{are } m\text{-graded.}\}\\
\cong\ &\mathrm{Aug}_m^a(Y';k)
\end{aligned}$$

where (C_i,d_i) is the complex over the vertical line $x=x_i$ (labeled by the dotted line i in Figure 3.2 (left)) determined by (d_0,r,s) via Lemma 3.4. That is, $(C_1,d_1) = s_b\circ H_s(C_0,d_0)$, $(C_R,d_R) = (C_2,d_2) = s_a\circ H_r(C_1,d_1)$. Similarly, given $(d_0,r,s)\in \mathrm{Aug}_m^a(Y;k)\cong \mathrm{Aug}_m^a(Y';k)$,

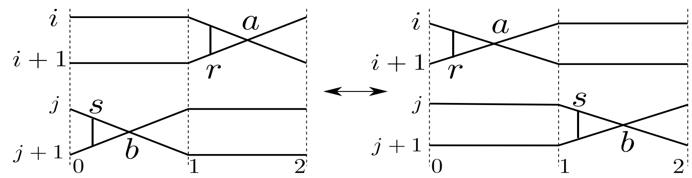

FIGURE 3.2. The move applied to modify MCSs, corresponding to a smooth isotopy which switches the x-coordinates of two neighboring crossings. In the figure, a, b are the crossings, and r, s indicate the coefficients of the handleslides in each diagram.

define $(C'_1, d'_1) := s_a \circ H_r(C_0, d_0)$, $(C_R, d'_R) = (C'_2, d'_2) := s_b \circ H_s(C'_1, d'_1)$ according to Figure 3.2 (right). Observe that $(s_a \circ H_r) \circ (s_b \circ H_s) = (s_b \circ H_s) \circ (s_a \circ H_r)$ as shown in Figure 3.2, so $(C_2, d_2) = (C'_2, d'_2)$.

Thus we obtain an isomorphism $\Phi_h : \mathrm{Aug}^a_m(Y; k) \xrightarrow{\sim} \mathrm{Aug}^a_m(Y'; k)$. Recall that, under the identification between augmentations and A-form MCSs, the restriction maps $r_L : \mathrm{Aug}_m(Y; k) \to \mathrm{Aug}_m(Y_L; k)$ (resp. $r_R : \mathrm{Aug}_m(Y; k) \to \mathrm{Aug}_m(Y_R; k)$) is given by $(d_0, r, s) \to (C_0, d_0)$ (resp. $(d_0, r, s) \to (C_2, d_2)$), and similarly for Y'. So, clearly the isomorphism Φ_h commutes with the identity map $Id : \mathrm{Aug}_m(Y_L; k) \xrightarrow{\sim} \mathrm{Aug}_m(Y'_L; k)$.

Moreover, as $(C_2, d_2) = (C'_2, d'_2)$, the isomorphism Φ_h also commutes with the identity map $\varphi_R = Id : \mathrm{Aug}_m(Y_R; k) \xrightarrow{\sim} \mathrm{Aug}_m(Y'_R; k)$. The theorem in this case then follows.

If h is a Legendrian Reidemeister type I move between Y, Y'. We may assume Y (resp. Y') is the Legendrian tangle as in Figure 3.3 (left) (resp. (right)) without the handleslides. In Figure 3.3, assume a is the crossing, and c is the marked right cusp, with marked point $*$ corresponding to a generator t in the DGA associated to Y. Denote by s_a elementary transformation corresponding to the crossing a. As always, label the strands over any generic vertical line $x = x_l$ from top to bottom by $1, 2, \ldots, n_l$. Denote by H_r, H_s the handleslides with coefficients $r, s \in k$ in Figure 3.3 to the immediate left of a, c respectively, denote by $H^{\uparrow}_r$ the *unfiltered handleslide* with coefficient $r \in k$ in Figure 3.4 (middle). Denote $C_L = C_0 = C(Y_L) = C(Y'_L), C_R = C(Y_R) = C(Y'_R)$ (Definition 1.21). Denote $\mu_i = \mu|_{Y|_{\{x=x_i\}}}$ for $0 \le i \le 2$, where the vertical line $x = x_i$ is labeled by the dotted line i in Figure 3.3 (left).

Under the identification between augmentations and A-form MCSs, we have:

$$\begin{aligned}\mathrm{Aug}^a_m(Y; k) \;\cong\; & \{(d_0, r, s) | (C_i, d_i) \text{is a } m\text{-graded filtered acyclic complex,} \\ & \text{and the handleslides } H_r, H_s \text{ are } m\text{-graded.}\}\end{aligned}$$

where (C_i, d_i) is the complex over the vertical line $x = x_i$ in Figure 3.3 (left) determined by (d_0, r, s) via Lemma 3.4. That is, $(C_1, d_1) = (C_0, d_0) \oplus \mathrm{Span}\{e_k, d_1 e_k = (-1)^{\mu_1(k)} e_{k+1}\}$ via the inclusion $C_0 \hookrightarrow C_1$ as in Definition 3.2 (5d), $(C_2, d_2) = s_a \circ H_r(C_1, d_1)$, and $(C_R, d_R) = (C_3, d_3) = Q_k \circ H_s(C_2, d_2)$, where Q_k is defined as in Definition 3.2 (5c). That is, $Q_k : (C, d) = H_s(C_2, d_2) \twoheadrightarrow (C_3, d_3)$ with $Q_k(e_i) = e_i$ for $i < k$, $Q_k(e_i) = e_{i-2}$ for $i > k+1$, and $Q_k(e_k) = 0 = Q_k(de_k)$.

Observe that, $< d_1 e_{k+1}, e_{k+2} >= 0$ is automatic, and $< d_2 e_k, e_{k+1} >=< d_1 e_k, e_{k+2} > -r < d_1 e_k, e_{k+1} >= -(-1)^{\mu_1(k)} r$, so r is the value assigned to the base point $*$ at c. Also, $|a| = 0$ implies

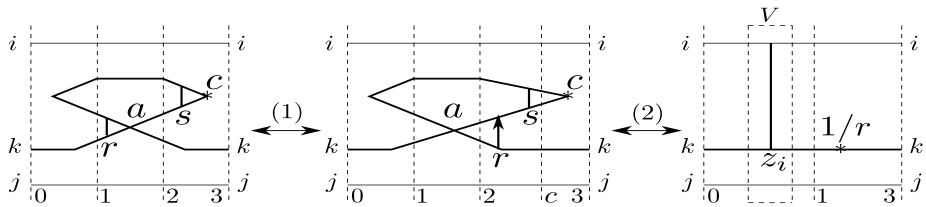

FIGURE 3.3. The sequence of moves applied to modify MCSs, corresponding to a Legendrian Reidemeister type I move. In the figure, a is the crossing, c is the marked right cusp, and $r \in k^*, s \in k$ indicate the coefficients of the corresponding (possibly unfiltered) handleslides. In the last diagram, $1/r$ indicates the coefficient of a labeled base point $*$, and V is a collection of handleslides: for each $i < k$, there's a handleslide between strands i, k with coefficient $z_i = (-1)^{\mu_2(k)} r^{-1} s < d_0 e_i, e_k >$, depending on the MCS (d_0, r, s).

that H_r is automatically m-graded. Hence equivalently, by Lemma 3.5, we have:

$$\begin{aligned} & \mathrm{Aug}_m^a(Y;k) \\ \cong\ & \{(d_0, r, s) | (C_0, d_0) \text{ is } m\text{-graded filtered and acyclic}, H_s \text{ is } m\text{-graded}, \\ & < d_1 e_{k+1}, e_{k+2} >= 0, < d_2 e_k, e_{k+1} >\neq 0.\} \\ \cong\ & \{(d_0, r, s) | (C_0, d_0) \text{ is } m\text{-graded filtered and acyclic}, H_s \text{ is } m\text{-graded}, r \in k^*\} \\ \cong\ & \mathrm{Aug}_m^a(Y';k) \times k^* \times k^\beta \end{aligned}$$

where $k^\beta \ni s$ encodes the possible values of s, with $\beta = 1$ (resp. 0 and $k^\beta = \{0\}$) if $m = 1$ (resp. $m \neq 1$). Thus, we obtain an isomorphism $\Phi_h : \mathrm{Aug}_m^a(Y;k) \xrightarrow{\sim} \mathrm{Aug}_m^a(Y';k) \times k^* \times k^\beta$ which sends (d_0, r, s) to (d_0, r, s). Clearly, Φ_h commutes with the identity map $Id : \mathrm{Aug}_m(Y_L;k) \xrightarrow{\sim} \mathrm{Aug}_m(Y'_L;k)$.

On the other hand, the right restriction maps are $r_R : \mathrm{Aug}_m^a(Y;k) \to \mathrm{Aug}_m(Y_R;k)$ (resp. $\mathrm{Aug}_m^a(Y';k) \times k^* \times k^\beta \to \mathrm{Aug}_m(Y'_R;k)$) given by $(d_0, r, s) \to (C_R, d_R) = (C_3, d_3)$ (resp. $(d_0, r, s) \to (C_R, d'_R) = (C_0, d_0)$). Observe that

$$\begin{aligned} (C_R, d_R) &= Q_k \circ H_s \circ s_a \circ H_r((C_0, d_0) \oplus \mathrm{Span}\{e_k, d_1 e_k = (-1)^{\mu_1(k)} e_{k+1}\}) \\ &= Q_k \circ H_s \circ H_r^{\uparrow} \circ s_a((C_0, d_0) \oplus \mathrm{Span}\{e_k, d'_1 e_k = (-1)^{\mu_1(k)} e_{k+1}\}) \\ &= Q_k \circ H_s \circ H_r^{\uparrow}((C_0, d_0) \oplus \mathrm{Span}\{e_k, d'_2 e_k = (-1)^{\mu_1(k)} e_{k+2}\}) \\ &= Q_k \circ H_s \circ H_r^{\uparrow}(C'_2, d'_2) \end{aligned}$$

as shown in Figure 3.3, where for $i = 1, 2$, (C'_i, d'_i) is the complex over the vertical line $x = x_i$ in Figure 3.3 (middle) determined by (d_0, r, s), i.e. $(C'_1, d'_1) = (C_0, d_0) \oplus \mathrm{Span}\{e_k, d'_1 e_k = (-1)^{\mu_1(k)} e_{k+1}\}$ via the inclusion $(C_0, d_0) \hookrightarrow (C'_1, d'_1)$ as in Definition 3.2 (5d), and $(C'_2, d'_2) = s_a(C'_1, d'_1) = (C_0, d_0) \oplus \mathrm{Span}\{e_k, d'_2 e_k = (-1)^{\mu_1(k)} e_{k+2}\}$, via the inclusion $(C_0, d_0) \hookrightarrow (C'_2, d'_2)$ given by: $e_i \to e_i$ for $i < k$; $e_k \to e_{k+1}$; $e_i \to e_{i+2}$ for $i > k$.

Given $(d_0, r, s) \in \mathrm{Aug}_m^a(Y;k)$, for each $i < k$, define $z_i := (-1)^{\mu_2(k)} r^{-1} s < d_0 e_i, e_k >$ and let $H_i(z_i)$ be the handleslide between strands i, k with coefficient z_i as in Figure 3.3 (right). Clearly, the handleslides $H_i(z_i)$'s commute with each other as elementary transformations. Let V be the collection of handleslides $H_i(z_i)$'s. Let c_y be the elementary transformation corresponding to the labeled base point with coefficient $y = r^{-1}$ as in Figure 3.3 (right). Define $\varphi_R := (c_y \circ V)^{-1} : C_R \xrightarrow{\sim} C_R$ depending on (d_0, r, s). Notice that if $z_i \neq 0$, then $s \neq 0$, so $m = 1$ by the condition $(d_0, r, s) \in \mathrm{Aug}_m^a(Y; k)$. Therefore, the handleslides $H_i(z_i)$'s are all m-graded. It follows that φ_R is a m-graded filtered isomorphism.

Claim: $(C_R, d_R) = \varphi_R^{-1}(C_R, d_R')(:= (C_R, \varphi_R^{-1} \circ d_R' \circ \varphi_R))$.

Proof of claim. Denote $Q := Q_k \circ H_s \circ H_r^{\uparrow} : C_2' \xrightarrow{\sim} C_R$. Let $(C_c, d_c) := H_s \circ H_r^{\uparrow}(C_2', d_2')$ be the complex over the vertical line immediately to the left of the right cusp c in Figure 3.3 (middle) (labeled by the dotted line c in the figure). Then Q_k is the surjective morphism $Q_k : (C_c, d_c) \to (C_R, d_R)$ with kernel $\mathrm{Span}\{e_k, d_c e_k\}$ defined as in Definition 3.2 (5c). Denote $\varphi_R^{-1} \cdot d_R' = \varphi_R^{-1} \circ d_R' \circ \varphi_R$.

Since $(C_2', d_2') = (C_0, d_0) \oplus \mathrm{Span}\{e_k, d_2' e_k = (-1)^{\mu_1(k)} e_{k+2}\}$, we have: for $i > k$, $d_R e_i = d_R Q(e_{i+2}) = Q d_2' e_{i+2} = d_0 e_i = (\varphi_R^{-1} \cdot d_R') e_i$; $d_R e_k = d_R Q(e_{k+2} + r e_{k+1}) = Q d_2'(e_{k+2} + r e_{k+1}) = r Q d_2' e_{k+1} = r d_0 e_k = (\varphi_R^{-1} \cdot d_R') e_k$, where we have used $d_2' e_{k+2} = 0$; for $i < k$, notice that $< d_2' e_i, e_k >= 0 =< d_2' e_i, e_{k+2} >$, we then have

$$\begin{aligned}
d_R e_i &= d_R Q(e_i) = Q d_2' e_i \\
&= Q(\sum_{i<l<k} < d_2' e_i, e_l > e_l + < d_2' e_i, e_{k+1} > e_{k+1} + \sum_{j>k} < d_2' e_i, e_{j+2} > e_{j+2}) \\
&= Q_k(\sum_{i<l<k} < d_0 e_i, e_l > e_l + < d_0 e_i, e_k > e_{k+1} + \sum_{j>k} < d_0 e_i, e_j > e_{j+2}) \\
&= \sum_{i<l<k} < d_0 e_i, e_l > e_l + < d_0 e_i, e_k > Q_k(e_{k+1}) + \sum_{j>k} < d_0 e_i, e_j > e_j
\end{aligned}$$

To compute $Q_k(e_{k+1})$, notice that

$$\begin{aligned}
d_c e_k &= d_c H_s H_r^{\uparrow}(e_k + s e_{k+1}) = H_s H_r^{\uparrow} d_2'(e_k + s e_{k+1}) \\
&= H_s H_r^{\uparrow}((-1)^{\mu_2(k)} e_{k+2} + s d_2' e_{k+1}) \\
&= H_s((-1)^{\mu_2(k)}(e_{k+2} - r e_{k+1}) + s d_2' e_{k+1}) \\
&= (-1)^{\mu_2(k)}(e_{k+2} - r e_{k+1}) + s d_2' e_{k+1}
\end{aligned}$$

It follows that

$$\begin{aligned}
e_{k+1} &= -(-1)^{\mu_2(k)} r^{-1} d_c e_k + r^{-1} e_{k+2} + (-1)^{\mu_2(k)} r^{-1} s d_2' e_{k+1} \\
&= -(-1)^{\mu_2(k)} r^{-1} d_c e_k + r^{-1} e_{k+2} + (-1)^{\mu_2(k)} r^{-1} s \sum_{j>k} < d_2' e_{k+1}, e_{j+2} > e_{j+2} \\
&= -(-1)^{\mu_2(k)} r^{-1} d_c e_k + r^{-1} e_{k+2} + (-1)^{\mu_2(k)} r^{-1} s \sum_{j>k} < d_0 e_k, e_j > e_{j+2}
\end{aligned}$$

which then implies that $Q_k(e_{k+1}) = r^{-1} e_k + (-1)^{\mu_2(k)} r^{-1} s \sum_{j>k} < d_0 e_k, e_j > e_j$. As a consequence, we obtain

$$d_R e_i = \sum_{i<l<k} < d_0 e_i, e_l > e_l + < d_0 e_i, e_k > r^{-1} e_k \tag{3.2.1}$$

$$
\begin{aligned}
& + \sum_{j>k}(< d_0e_i, e_j > +(-1)^{\mu_2(k)}r^{-1}s < d_0e_i, e_k >< d_0e_k, e_j >)e_j \\
& = \sum_{i<l<k} < d_0e_i, e_l > e_l + < d_0e_i, e_k > r^{-1}e_k \\
& + \sum_{j>k}(< d_0e_i, e_j > +z_i < d_0e_k, e_j >)e_j
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(\varphi_R^{-1} \cdot d_R')(e_i) &= \varphi_R^{-1} \circ d_R' \circ \varphi_R(e_i) \\
&= \varphi_R^{-1} \circ d_R'(e_i + z_ie_k) = c_y \circ V(d_0e_i + z_id_0e_k) \\
&= c_y(\sum_{i<l<k} < d_0e_i, e_l > (e_l - z_le_k) + < d_0e_i, e_k > e_k) \\
&+ c_y(\sum_{j>k}(< d_0e_i, e_j > +z_i < d_0e_k, e_j >)e_j) \\
&= \sum_{i<l<k} < d_0e_i, e_l > (e_l - z_lye_k) + < d_0e_i, e_k > ye_k \\
&+ \sum_{j>k}(< d_0e_i, e_j > +z_i < d_0e_k, e_j >)e_j \\
&= \sum_{i<l<k} < d_0e_i, e_l > e_l + < d_0e_i, e_k > r^{-1}e_k \\
&+ \sum_{j>k}(< d_0e_i, e_j > +z_i < d_0e_k, e_j >)e_j \\
&= d_Re_i
\end{aligned} \tag{3.2.2}
$$

where in the second to the last identity, we've used:

$$
\begin{aligned}
\sum_{i<l<k} < d_0e_i, e_l > z_l &= (-1)^{\mu_2(k)}r^{-1}s \sum_{i<l<k} < d_0e_i, e_l >< d_0e_l, e_k > \\
&= (-1)^{\mu_2(k)}r^{-1}s < d_0^2e_i, e_k >= 0.
\end{aligned}
$$

Now we have seen that $d_Re_i = (\varphi_R^{-1} \cdot d_R')e_i$ for all i. This finishes the proof of the claim. □

It follows from the claim that, $\varphi_R : (C_R, d_R) \xrightarrow{\sim} (C_R, d_R')$ is an $\mathbb{Z}/m$-graded filtered isomorphism. In particular, (C_R, d_R) and (C_R, d_R') induce the same m-graded normal ruling of $Y_R = Y_R'$. Hence, after passing to the left and right pieces $Y_L = Y_L', Y_R = Y_R'$, the isomorphism Φ_h is compatible with the ruling decomposition. In other words, $(\phi_h \circ R_Y(d_0, r, s))|_{Y_L'} = (R_{Y'} \circ \Phi_h(d_0, , r, s))|_{Y_L'}$ and $(\phi_h \circ R_Y(d_0, r, s))|_{Y_R'} = (R_{Y'} \circ \Phi_h(d_0, r, s))|_{Y_R'}$ for all (d_0, r, s) in $\mathrm{Aug}_m^a(Y; k)$. The theorem in this case follows.

Moreover, notice that any m-graded normal ruling of Y (resp. Y') is uniquely determined by its restrictions to Y_L, Y_R (resp. Y_L', Y_R'), and so is ϕ_h. It follows that Φ_h in this case is fact compatible with $\phi_h : \mathrm{NR}_Y^m \xrightarrow{\sim} \mathrm{NR}_{Y'}^m$.

If h is a Legendrian Reidemeister type II move involving a right cusp between Y, Y'. We may assume Y (resp. Y') is the Legendrian tangle as in Figure 3.4 (left) (resp. (right)) without the handleslides. In Figure 3.4, assume a, b are the crossings and c is the right cusp, and denote by s_a, s_b the corresponding elementary transformations. As always, label the strands over any generic vertical line $x = x_l$ from top to bottom by $1, 2, \ldots, n_l$. Denote by H_r, H_s, H_t the handleslides with coefficients $r, s, t \in k$ in Figure 3.4 (left), and similarly denote by H_r, H'_s, H'_t the corresponding handleslides in Figure 3.4 (middle and right). Denote $C_L = C_0 = C(Y'_L) = C(Y_L), C_R = C(Y'_R) = C(Y_R)$ (Definition 1.21). Denote $\mu_L := \mu|_{Y_L = Y'_L}$.

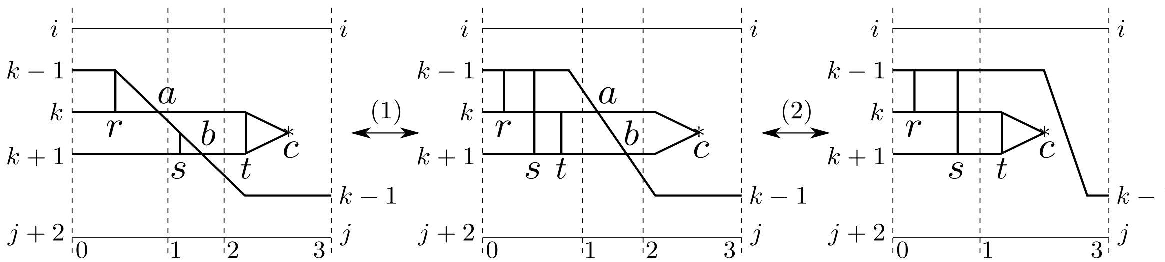

FIGURE 3.4. The sequence of moves applied to modify MCSs, corresponding to a Legendrian Reidemeister type II move involving a right cusp. Label the strands over any generic vertical line from top to bottom by $1, 2, \ldots$. In the figure, a, b are the crossings, c is the right cusp, and r, s, t indicate the corresponding handleslides with coefficients $r, s, t \in k$ respectively.

Under the identification between the augmentations and A-form MCSs, we then have:

$$\begin{aligned} \mathrm{Aug}^a_m(Y; k) \;\cong\; & \{(d_0, r, s, t) | (C_i, d_i) \text{is a } m\text{-graded filtered acyclic complex,} \\ & \text{and the handelslides } H_r, H_s, H_t \text{ are } m\text{-graded.}\} \end{aligned}$$

where (C_i, d_i) is the complex over the vertical line $x = x_i$ (labeled by the dotted line i in Figure 3.4 (left)) determined by (d_0, r, s, t) via Lemma 3.4. That is, $(C_1, d_1) = s_a \circ H_r(C_0, d_0)$, $(C_2, d_2) = s_b \circ H_s(C_1, d_1)$, and $(C_R, d_R) = (C_3, d_3) = Q_{k-1} \circ H_t(C_2, d_2)$, where $Q_{k-1} = Q_{k-1}(H_t \cdot d_2)$ is the morphism φ in Definition 3.2 (5c). In other words, we have a short exact sequence of $\mathbb{Z}/m$-graded filtered complexes:

$$0 \to \mathrm{Span}\{e_{k-1} + te_k, d_2(e_{k-1} + te_k)\} \to (C_2, d_2) \xrightarrow{Q_{k-1} \circ H_t} (C_R, d_R) \to 0 \tag{3.2.3}$$

Notice that $< d_1 e_k, e_{k+1} >=< d_0 e_{k-1}, e_{k+1} > +r < d_0 e_k, e_{k+1} >, < d_2 e_k, e_{k+1} >=< d_0 e_k, e_{k+1} >$. Then equivalently, by Lemma 3.5, we have:

$$\begin{aligned} & \mathrm{Aug}^a_m(Y; k) \\ \cong\; & \{(d_0, r, s, t) | (C_0, d_0) \text{ is } m\text{-graded filtered acyclic}, H_r, H_s, H_t \text{ are } m\text{-graded,} \\ & < d_0 e_{k-1}, e_k >= 0, < d_0 e_{k-1}, e_{k+1} > +r < d_0 e_k, e_{k+1} >= 0, < d_0 e_k, e_{k+1} >\neq 0.\} \\ \cong\; & \{(d_0, s, t) | (C_0, d_0) \text{ is } m\text{-graded filtered acyclic}, H_s, H_t \text{ are } m\text{-graded,} \\ & < d_0 e_k, e_{k+1} >\neq 0.\} \\ \cong\; & \{(d_0, t) | (C_0, d_0) \text{ is } m\text{-graded filtered acyclic}, H_t \text{ is } m\text{-graded,} \\ & < d_0 e_k, e_{k+1} >\neq 0.\} \times k^\beta \end{aligned}$$

$$\cong \quad \mathrm{Aug}^a_m(Y';k)\times k^\beta$$

where in the second identification we have observed that: $< d_0e_{k-1}, e_k >= 0$ follows automatically from $< d_0^2 e_{k-1}, e_{k+1} >= 0$ and $< d_0e_k, e_{k+1} >\neq 0$; The condition $< d_0e_{k-1}, e_{k+1} > +r < d_0e_k, e_{k+1} >= 0$ implies $r = -< d_0e_{k-1}, e_{k+1} > / < d_0e_k, e_{k+1} >$, which is nonzero only when $\mu_L(k-1) = \mu_L(k+1)+1(=\mu_L(k))(\mathrm{mod} m)$, i.e. $|a| = \mu_L(k-1)-\mu_L(k) = 0(\mathrm{mod} m)$, or equivalently H_r is m-graded. The last identification again follows from Lemma 3.5, where $k^\beta \ni s$ encodes the possible values of s, with $\beta = 1$ (resp. 0 and $k^\beta = \{0\}$) if $|b| = 0(\mathrm{mod} m)$ (resp. $|b| \neq 0(\mathrm{mod} m)$).

Thus, we obtain an isomorphism $\Phi_h : \mathrm{Aug}^a_m(Y;k) \xrightarrow{\sim} \mathrm{Aug}^a_m(Y';k)\times k^\beta$ which sends (d_0, r, s, t) to (d_0, t, s) with $r = -< d_0e_{k-1}, e_{k+1} > / < d_0e_k, e_{k+1} >$. Recall that, under the identification between augmentations and A-form MCSs, the left restriction maps are $r_L : \mathrm{Aug}^a_m(Y;k) \to \mathrm{Aug}_m(Y_L;k)$ (resp. $\mathrm{Aug}^a_m(Y';k) \times k^\beta \to \mathrm{Aug}_m(Y'_L;k)$) given by $(d_0, r, s, t) \to (C_0, d_0)$ (resp. $(d_0, t, s) \to (C_0, d_0)$), and the right restriction maps are $r_R : \mathrm{Aug}^a_m(Y;k) \to \mathrm{Aug}_m(Y_R;k)$ (resp. $\mathrm{Aug}^a_m(Y';k) \times k^\beta \to \mathrm{Aug}_m(Y'_R;k)$) given by $(d_0, r, s, t) \to (C_R, d_R)$ (resp. $(d_0, t, s) \to (C_R, d'_R)$), where $(C_R, d_R) = (C_3, d_3)$ (resp. $(C_R, d'_R) := Q_k \circ H'_t(C_0, d_0)$). Clearly, Φ_h commutes with the identity map $Id : \mathrm{Aug}_m(Y_L;k) \xrightarrow{\sim} \mathrm{Aug}_m(Y'_L;k)$.

On the other hand, given $(d_0, r, s, t) \in \mathrm{Aug}^a_m(Y;k) \cong \mathrm{Aug}^a_m(Y';k)\times k^\beta$, so $r = -< d_0e_{k-1}, e_{k+1} > / < d_0e_k, e_{k+1} >$ and H_r, H_s, H_t are m-graded, observe that

$$\begin{aligned}(C_R, d_R) &= Q_{k-1} \circ H_t \circ s_b \circ H_s \circ s_a \circ H_r(C_0, d_0)\\ &= Q_{k-1} \circ s_b \circ s_a \circ H'_t \circ H'_s \circ H_r(C_0, d_0)\\ &= Q_k \circ H'_t \circ H'_s \circ H_r(C_0, d_0)\end{aligned}$$

as shown in Figure 3.4, where the first move (1) in Figure 3.4 is a sequence of *Type 1* Handleslide moves (Section 3.1.3) corresponding the second identity above, the second move (2) in Figure 3.4 corresponds to the last equality above, and can be verified by a direct calculation.

Let $(C_0, d'_0) := H'_s \circ H_r(C_0, d_0)$. Observe that $H'_s \circ H_r$ is $\mathbb{Z}/m$-graded, filtration preserving, and preserves the sub-complex $\mathrm{Span}\{e_k + te_{k+1}, d_0(e_k + te_{k+1})\}$ of (C_0, d_0), we then obtain an isomorphism of short exact sequences of $\mathbb{Z}/m$-graded filtered complexes:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Span}\{e_k + te_{k+1}, d_0(e_k + te_{k+1})\} \arrow[r] \arrow[d, "\cong"] & (C_0, d_0) \arrow[r, "Q_k\circ H'_t"] \arrow[d, "H'_s\circ H_r"] & (C_R, d'_R) \arrow[r] \arrow[d, dashed, "\varphi_R^{-1}"] & 0 \\ 0 \arrow[r] & \mathrm{Span}\{e_k + te_{k+1}, d'_0(e_k + te_{k+1}\} \arrow[r] & (C_0, d'_0) \arrow[r, "Q_k\circ H'_t"] & (C_R, d_R) \arrow[r] & 0 \end{tikzcd}$$

In particular, we obtain an induced isomorphism $\varphi_R : (C_R, d_R) \xrightarrow{\sim} (C_R, d'_R)$ of $\mathbb{Z}/m$-graded filtered complexes, which then induces the same m-graded normal ruling of $Y_R = Y'_R$. Hence, after passing to the left and right pieces $Y_L = Y'_L, Y_R = Y'_R$, the isomorphism Φ_h is compatible with the ruling decomposition. The theorem in this case follows.

Moreover, notice that any m-graded normal ruling of Y (resp. Y') is uniquely determined by its restrictions to Y_L, Y_R (resp. Y'_L, Y'_R), and so is ϕ_h. It follows that Φ_h is in fact compatible with $\phi_h : \mathrm{NR}^m_Y \xrightarrow{\sim} \mathrm{NR}^m_{Y'}$.

If h is a Legendrian Reidemeister type II move involving a left cusp between Y, Y'. We may assume Y (resp. Y') is the Legendrian tangle as in Figure 3.5 (left) (resp. (right)) without the handleslides. In Figure 3.5, assume a, b are the crossings, and denote by s_a, s_b the corresponding elementary transformations. As always, label the strands over any generic vertical line $x = x_l$ from top to bottom by $1, 2, \ldots, n_l$. Denote by H_r the handleslide with coefficient $r \in k$ in Figure 3.5 (left), and similarly denote by $H_r^{\uparrow}$ the *unfilterd handleslide* in Figure 3.5 (second and third). Denote $C_L = C_0 = C(Y_L) = C(Y'_L), C_R = C(Y_R) = C(Y'_R)$ (Definition 1.21). Denote $\mu_L := \mu|_{Y_L=Y'_L}, \mu_R := \mu|_{Y_R=Y'_R}, \mu_1 = \mu|_{Y|_{\{x=x_1\}}}$.

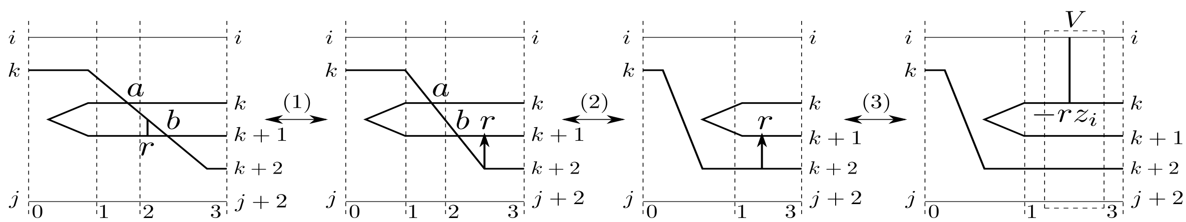

FIGURE 3.5. The sequence of moves applied to modify MCSs, corresponding to a Legendrian Reidemeister type II move involving a left cusp. In the figure, a, b are the crossings, r indicates the coefficient of the handleslide (resp. unfiltered handleslides) in the first diagram (resp. the second and third diagrams). In the last diagram, V is a collection of handleslides: for each $i < k$, there's a handleslide between strands i, k with coefficient $-rz_i$, where $z_i = (-1)^{\mu_R(k)} < d_0 e_i, e_k >$, with (C_0, d_0) the initial complex in the MCS.

Notice that, for any augmentation $\epsilon \in \mathrm{Aug}_m(Y; k)$, have $\epsilon \circ \partial b = (-1)^{|a|+1}\epsilon(a) = 0$, equivalently, the possible handleslide immediately to the left of a is trivial in the corresponding A-form MCS. Hence, under the identification between augmentations and A-form MCSs, we have:

$$\begin{aligned} \mathrm{Aug}_m^a(Y; k) \;&\cong\; \{(d_0, r) | (C_i, d_i) \text{is a } m\text{-graded filtered acyclic complex,} \\ &\quad\;\; \text{and the handelslide } H_r \text{ is } m\text{-graded.}\} \end{aligned}$$

where (C_i, d_i) is the complex over the vertical line $x = x_i$ (labeled by the dotted line i in Figure 3.5 (left)) determined by (d_0, r) via Lemma 3.4. That is, $(C_1, d_1) = (C_0, d_0) \oplus \mathrm{Span}\{e_{k+1}, d_1 e_{k+1} = (-1)^{\mu_1(k+1)} e_{k+2}\}$ as in Definition 3.2 (5d), $(C_2, d_2) = s_a(C_1, d_1)$, and $(C_R, d_R) = (C_3, d_3) = s_b \circ H_r(C_2, d_2)$. Equivalently, by Lemma 3.5, we have:

$$\begin{aligned} &\mathrm{Aug}_m^a(Y; k) \\ \cong\; &\{(d_0, r) | (C_0, d_0) \text{ is } m\text{-graded filtered and acyclic}, H_r \text{ is } m\text{-graded.}\} \\ \cong\; &\mathrm{Aug}_m^a(Y'; k) \times k^\beta \end{aligned}$$

where $k^\beta \ni r$ encodes the possible values of r, with $\beta = 1$ (resp. 0 and $k^\beta = \{0\}$) if $|b| = 0(\mathrm{mod} m)$ (resp. $|b| \neq 0(\mathrm{mod} m)$). Thus, we obtain an isomorphism $\Phi_h : \mathrm{Aug}_m^a(Y; k) \xrightarrow{\sim} \mathrm{Aug}_m^a(Y'; k) \times k^\beta$ which sends (d_0, r) to (d_0, r). Clearly, Φ_h commutes with the identity map $Id : \mathrm{Aug}_m(Y_L; k) \xrightarrow{\sim} \mathrm{Aug}_m(Y'_L; k)$.

On the other hand, the right restriction maps are $r_R : \mathrm{Aug}_m^a(Y; k) \to \mathrm{Aug}_m(Y_R; k)$ (resp. $\mathrm{Aug}_m^a(Y'; k) \times k^\beta \to \mathrm{Aug}_m(Y'_R; k)$) given by $(d_0, r) \to (C_R, d_R)$ (resp. (C_R, d'_R)), where $(C_R, d_R) = (C_3, d_3)$ (resp. $(C_R, d'_R) := (C_0, d_0) \oplus \mathrm{Span}\{e_k, d_R e_k = (-1)^{\mu_R(k)} e_{k+1}\}$ as in Definition 3.2 (5d)).

Observe that

$$\begin{aligned}(C_R,d_R) &= s_b\circ H_r\circ s_a((C_0,d_0)\oplus\mathrm{Span}\{e_{k+1},d_1e_{k+1}=(-1)^{\mu_1(k+1)}e_{k+2}\})\\ &= H_r^{\uparrow}\circ s_b\circ s_a((C_0,d_0)\oplus\mathrm{Span}\{e_{k+1},d_1e_{k+1}=(-1)^{\mu_R(k)}e_{k+2}\})\\ &= H_r^{\uparrow}((C_0,d_0)\oplus\mathrm{Span}\{e_k,d_1'e_k=(-1)^{\mu_R(k)}e_{k+1}\})\\ &\cong H_r^{\uparrow}(C_R,d_R')\end{aligned}$$

as shown in Figure 3.5, where (C_1',d_1') is the complex over $x=x_1$ in Figure 3.5 (third) determined by (C_0,d_0), i.e. $(C_1',d_1')=(C_0,d_0)\oplus\mathrm{Span}\{e_k,d_1'e_k=(-1)^{\mu_R(k)}e_{k+1}\}$ as in Definition 3.2 (5d), which can also be identified with (C_R,d_R').

For each $i<k$, let $z_i:=(-1)^{\mu_R(k)}<d_R'e_i,e_{k+2}>=(-1)^{\mu_R(k)}<d_0e_i,e_k>$ and let $H_i(-rz_i)$ be the handleslide between strands i,k with coefficient $-rz_i$ as in Figure 3.5 (right). Clearly, as elementary transformations, they commute with each other (See Figure 3.1 (b)). Let $V=V(d_0,r)$ be the collection of handleslides $H_i(-rz_i)$ shown as in Figure 3.5 (right), which also represents an $\mathbb{Z}/m$-graded filtered isomorphism $V:C_R\xrightarrow{\sim}C_R$, where we have identified C_1' with C_R.

Claim: $H_r^{\uparrow}(C_R,d_R')=V(C_R,d_R')$.

Proof of claim. In fact, for $i\geq k$ have $<d_R'e_i,e_{k+2}>=0$ and $d_R'e_{k+1}=0$, hence $(H_r^{\uparrow}\cdot d_R')(e_i)=H_r^{\uparrow}\circ d_R'\circ(H_r^{\uparrow})^{-1}(e_i)=d_R'e_i=(V\cdot d_R')(e_i)$; For $i<k$ have $(H_r^{\uparrow}\cdot d_R')(e_i)=d_R'e_i-r<d_R'e_i,e_{k+2}>e_{k+1}=d_R'e_i-rz_id_R'e_k$. On the other hand, have

$$\begin{aligned}(V\cdot d_R')(e_i) &= V\circ d_R'\circ V^{-1}(e_i)=V\circ d_R'(e_i-rz_ie_k)\\ &= V(d_R'e_i-rz_i(-1)^{\mu_R(k)}e_{k+1})\\ &= d_R'e_i+\sum_{i<l<k}<d_R'e_i,e_l>rz_le_k-rz_i(-1)^{\mu_R(k)}e_{k+1}\\ &= d_R'e_i-rz_i(-1)^{\mu_R(k)}e_{k+1}\\ &= (H_r^{\uparrow}\cdot d_R')(e_i)\end{aligned}$$

where we have observed that

$$\begin{aligned}\sum_{i<l<k}<d_R'e_i,e_l>rz_l &= (-1)^{\mu_R(k)}r\sum_{i<l<k}<d_R'e_i,e_l><d_R'e_l,e_{k+2}>\\ &= (-1)^{\mu_R(k)}r\sum_{i<l<k+2}<d_R'e_i,e_l><d_R'e_l,e_{k+2}>\\ &= (-1)^{\mu_R(k)}r<d_R'^2e_i,e_{k+2}>=0\end{aligned}$$

as $<d_R'e_k,e_{k+2}>=0=<d_R'e_{k+1},e_{k+2}>$. Therefore, $H_r^{\uparrow}\cdot d_R'=V\cdot d_R'$ as desired. □

As a consequence, $\varphi_R=V^{-1}:(C_R,d_R)\xrightarrow{\sim}(C_R,d_R')$ is an $\mathbb{Z}/m$-graded filtered isomorphism. In particular, (C_R,d_R) and (C_R,d_R') induce the same m-graded normal ruling of $Y_R=Y_R'$. Hence, after passing to the left and right pieces $Y_L=Y_L',Y_R=Y_R'$, the isomorphism Φ_h is compatible with the ruling decomposition. The theorem in this case follows.

Moreover, notice that any m-graded normal ruling of Y (resp. Y') is uniquely determined by its restrictions to Y_L, Y_R (resp. Y'_L, Y'_R), and so is ϕ_h. It follows that Φ_h is in fact compatible with $\phi_h : \mathrm{NR}^m_Y \xrightarrow{\sim} \mathrm{NR}^m_{Y'}$.

If h is a Legendrian Reidemeister type III move between Y, Y'. We may assume Y (resp. Y') is the Legendrian tangle as in Figure 3.6 (left) (resp. (right)) without the handleslides. In Figure 3.6, assume a, b, c are the crossings, and denote by s_a, s_b, s_c (resp. s'_a, s'_b, s'_c) the corresponding elementary transformations in Figure 3.6 (first and second diagrams) (resp. (third and fourth diagrams)). As always, label the strands over any generic vertical line $x = x_l$ from top to bottom by $1, 2, \ldots, n_l$. Denote by H_r, H_s, H_t (resp. $H'_r, H'_s, H'_{t'}$) the handleslides with coefficients $r, s, t \in k$ (resp. $r, s, t' = t - rs \in k$) in Figure 3.6 (left) (resp. (right)). Denote by $H_{r,2}, H_{s,2}, H_{t,2}$ (resp. $H_{r,3}, H_{s,3}, H_{t',3}$) the handleslides with coefficients $r, s, t \in k$ (resp. $r, s, t' = t - rs \in k$) in Figure 3.6 (second diagram) (resp. (third diagram)). Denote $C_L = C_0 = C(Y_L) = C(Y'_L), C_R = C(Y_R) = C(Y'_R)$ (Definition 1.21). Denote $\mu_L := \mu|_{Y_L = Y'_L}, \mu_R := \mu|_{Y_R = Y'_R}, \mu_1 = \mu|_{Y|_{\{x=x_1\}}}$.

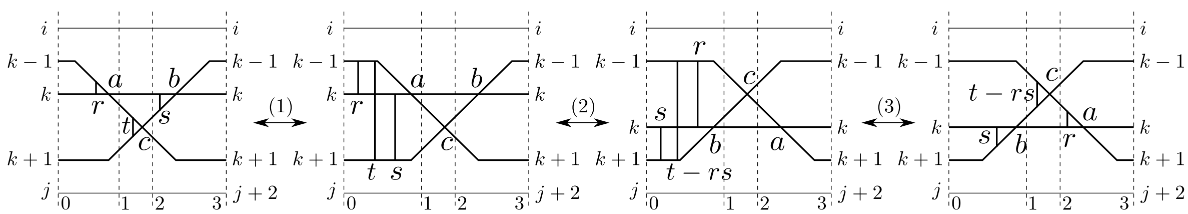

FIGURE 3.6. The sequence of Handleslide moves applied to modify MCSs, corresponding to a Legendrian Reidemeister type III move. In the figure, a, b, c are the crossings, $r, s, t, t' = t - rs$ indicate the coefficients of the handleslides in each diagram. The moves $(1), (3)$ are Handleslide *type 1 moves* (Figure 3.1 (c),(f)), and the move (2) is combination of Handleslide *type 2 moves*, *type 3 move* (Figure 3.1 (d),(b),(a)), and the standard identity $s_b \circ s_c \circ s_a = s'_a \circ s'_c \circ s'_b$.

Under the identification between augmentations and A-form MCSs, we have:

$$\begin{aligned} \mathrm{Aug}^a_m(Y;k) \quad \cong \quad & \{(d_0, r, s, t) | (C_i, d_i) \text{is a } m\text{-graded filtered acyclic complex,} \\ & \text{and the handleslides } H_r, H_s, H_t \text{ are } m\text{-graded.}\} \end{aligned}$$

where (C_i, d_i) is the complex over the vertical line $x = x_i$ (labeled by the dotted line i in Figure 3.6 (left)) determined by (d_0, r, s, t) via Lemma 3.4. That is, $(C_1, d_1) = s_a \circ H_r(C_0, d_0)$, $(C_2, d_2) = s_c \circ H_t(C_1, d_1)$, and $(C_R, d_R) = (C_3, d_3) = s_b \circ H_s(C_2, d_2)$. Observe that $< d_1 e_k, e_{k+1} >=< d_0 e_{k-1}, e_{k+1} > +r < d_0 e_k, e_{k+1} >$, and $< d_2 e_{k-1}, e_k >=< d_1 e_{k-1}, e_{k+1} > -t < d_1 e_{k-1}, e_k >=< d_0 e_k, e_{k+1} >$ as $< d_1 e_{k-1}, e_k >= 0$. Also, it's direct to see that H_r, H_s, H_t are all m-graded if and only if $H'_r, H'_s, H'_{t'}$ are all m-graded. Then equivalently, by Lemma 3.5, we have:

$$\begin{aligned} & \mathrm{Aug}^a_m(Y;k) \\ \cong \quad & \{(d_0, r, s, t) | (C_0, d_0) \text{ is } m\text{-graded filtered and acyclic}, H_r, H_s, H_t \text{ are } m\text{-graded}, \\ & < d_0 e_{k-1}, e_k >= 0, < d_1 e_k, e_{k+1} >= 0, < d_2 e_{k-1}, e_k >= 0.\} \\ \cong \quad & \{(d_0, r, s, t') | (C_0, d_0) \text{ is } m\text{-graded filtered and acyclic}, H'_r, H'_s, H'_{t'} \text{ are } m\text{-graded,} \end{aligned}$$

$$
\begin{aligned}
& \quad < d_0 e_{k-1}, e_k >= 0, < d_0 e_k, e_{k+1} >= 0, < d_0 e_{k-1}, e_{k+1} >= 0.\} \\
\cong & \quad \mathrm{Aug}_m^a(Y'; k)
\end{aligned}
$$

where the last identification follows by symmetry. Thus, we obtain an isomorphism Φ_h : $\mathrm{Aug}_m^a(Y;k) \xrightarrow{\sim} \mathrm{Aug}_m^a(Y';k)$ which sends (d_0, r, s, t) to (d_0, r, s, t') with $t' = t - rs$. Clearly, Φ_h commutes with the identity map $Id : \mathrm{Aug}_m(Y_L;k) \xrightarrow{\sim} \mathrm{Aug}_m(Y'_L;k)$.

On the other hand, the right restriction maps are $r_R : \mathrm{Aug}_m^a(Y;k) \to \mathrm{Aug}_m(Y_R;k)$ (resp. $\mathrm{Aug}_m^a(Y';k) \to \mathrm{Aug}_m(Y'_R;k)$) given by $(d_0, r, s, t) \to (C_R, d_R)$ (resp. $(d_0, r, s, t') \to (C_R, d'_R)$), where $(C_R, d_R) = (C_3, d_3)$ (resp. $(C_R, d'_R) = (C'_3, d'_3)$). Here (C'_i, d'_i) is the complex over the vertical line $x = x_i$ (labeled by the dotted line i in Figure 3.6 (right)), determined by (d_0, r, s, t') via Lemma 3.4. That is, $(C'_1, d'_1) = s'_b \circ H'_s(C_0, d_0)$, $(C'_2, d'_2) = s'_c \circ H'_{t'}(C'_1, d'_1)$, and $(C'_3, d'_3) = s'_a \circ H'_r(C'_2, d'_2)$. Observe that

$$
\begin{aligned}
(C_R, d_R) &= s_b \circ H_s \circ s_c \circ H_t \circ s_a \circ H_r(C_0, d_0) \\
&= s_b \circ s_c \circ s_a \circ H_{s,2} \circ H_{t,2} \circ H_{r,2}(C_0, d_0) \\
&= s'_a \circ s'_c \circ s'_b \circ H_{r,3} \circ H_{t',3} \circ H_{s,3}(C_0, d_0) \\
&= s'_a \circ H'_r \circ s'_c \circ H'_{t'} \circ s'_b \circ H'_s(C_0, d_0) \\
&= (C_R, d'_R)
\end{aligned}
$$

as shown in Figure 3.6. As a consequence, $\varphi_R = Id : (C_R, d_R) \xrightarrow{\sim} (C_R, d'_R)$ is an $\mathbb{Z}/m$-graded filtered isomorphism. In particular, (C_R, d_R) and (C_R, d'_R) induce the same m-graded normal ruling of $Y_R = Y'_R$.

Hence, we have seen that, after passing to the left and right pieces $Y_L = Y'_L, Y_R = Y'_R$, the isomorphism Φ_h is compatible with the ruling decomposition. The theorem in this case follows.

Moreover, notice that any m-graded normal ruling of Y (resp. Y') is uniquely determined by its restrictions to Y_L, Y_R (resp. Y'_L, Y'_R), and so is ϕ_h. It follows that Φ_h is in fact compatible with $\phi_h : \mathrm{NR}_Y^m \xrightarrow{\sim} \mathrm{NR}_{Y'}^m$.

So far, we have shown the theorem for Y, Y', hence there's an isomorphism $\Phi_{h,Y} : \mathrm{Aug}_m^a(Y;k) \times (k^*)^\alpha \times k^\beta \xrightarrow{\sim} \mathrm{Aug}_m^a(Y';k) \times (k^*)^{\alpha'} \times k^{\beta'}$, which commutes with $Id : \mathrm{Aug}_m^a(Y_L;k) \xrightarrow{\sim} \mathrm{Aug}_m^a(Y'_L;k)$, and is compatible with the ruling decomposition over $Y_R = Y'_R$. In general, $T = X \circ Y \circ Z, T' = X \circ Y' \circ Z$ are compositions of simpler Legendrian tangles, and h is the simple Legendrian isotopy between Y, Y'. In particular, $X_R = Y_L = Y'_L, Y_R = Y'_R = Z_L$. We construct an isomorphism $\Phi_h : \mathrm{Aug}_m^a(T;k) \times (k^*)^\alpha \times k^\beta \xrightarrow{\sim} \mathrm{Aug}_m^a(T';k) \times (k^*)^{\alpha'} \times k^{\beta'}$ as follows.

By the sheaf property, we have

$$
\begin{aligned}
\mathrm{Aug}_m^a(T;k) &= \mathrm{Aug}_m^a(X;k) \times_{\mathrm{Aug}_m^a(Y_L;k)} \times \mathrm{Aug}_m^a(Y;k) \times_{\mathrm{Aug}_m^a(Y_R;k)} \times \mathrm{Aug}_m^a(Z;k) \\
\mathrm{Aug}_m^a(T';k) &= \mathrm{Aug}_m^a(X;k) \times_{\mathrm{Aug}_m^a(Y'_L;k)} \times \mathrm{Aug}_m^a(Y';k) \times_{\mathrm{Aug}_m^a(Y'_R;k)} \times \mathrm{Aug}_m^a(Z;k)
\end{aligned}
$$

By the identification between augmentations and A-form MCSs, any element of $\mathrm{Aug}_m^a(T;k) \times (k^*)^\alpha \times k^\beta$ is of the form $(C_X, C_Y, \underline{r}, \underline{s}, C_Z)$, where C_X, C_Y, C_Z is an *acyclic* m-graded A-form MCS on X, Y, Z respectively, such that $C_X|_{Y_L} = C_Y|_{Y_L}, C_Y|_{Y_R} = C_Z|_{Y_R}$ as $\mathbb{Z}/m$-graded filtered complexes, and $\underline{r} \in (k^*)^\alpha$, $\underline{s} \in k^\beta$. Also, there's a similar statement for T'.

Define $(C_{Y'}, \underline{r'}, \underline{s'}) := \Phi_{h,Y}(C_Y, \underline{r}, \underline{s})$. Since $\Phi_{h,Y}$ commutes with $Id : \mathrm{Aug}_m^a(Y_L; k) \xrightarrow{\sim} \mathrm{Aug}_m^a(Y'_L; k)$, we have $C_X|_{Y'_L} = C_Y|_{Y_L} = C_{Y'}|_{Y'_L}$. On the other hand, the proof of the theorem for Y, Y' also gives a canonical $\mathbb{Z}/m$-graded filtered isomorphism $\varphi_R : C_Z|_{Z_L} = C_Y|_{Y_R} \xrightarrow{\sim} C_{Y'}|_{Y'_R = Y_R}$, with φ_R depending on $(C_Y, \underline{r}, \underline{s})$ algebraically. By Definition 3.2, the A-form MCS C_Z on Z has the form $C_Z = (\{(C_l, d_l)\}, \{x_l\}, H)$, where x_0 is the x-coordinate of the left endpoints of Z, $(C_0, d_0) = C_Z|_{Z_L}$, etc. Then by Lemma 3.14 below, there's a canonical extension of $(C_0, d'_0) = C_{Y'}|_{Y'_R}$ to a m-graded A-form MCS $C'_Z = (\{(C_l, d'_l)\}, \{x_l\}, H') := \Phi(\varphi_R, C_Z)$, and $\phi_0 = \varphi_R$ to a sequence of isomorphisms $\phi_Z =: \phi = \{\phi_l\}$, with $\phi_l : (C_l, d_l) \xrightarrow{\sim} (C_l, d'_l)$ an isomorphism of $\mathbb{Z}/m$-graded filtered complexes. Moreover, C'_Z and ϕ_Z depend on $C_Z|_{Z_L}, \varphi_R$ algebraically. As a consequence, $(C_X, C_{Y'}, \underline{r'}, \underline{s'}, C'_Z)$ is an element in $\mathrm{Aug}_m^a(T'; k) \times (k^*)^{\alpha'} \times k^{\beta'}$. *define* $\Phi_h(C_X, C_Y, \underline{r}, \underline{s}, C_Z) := (C_X, C_{Y'}, \underline{r'}, \underline{s'}, C'_Z)$. We then obtain an algebraic morphism

$$\Phi_h : \mathrm{Aug}_m^a(T; k) \times (k^*)^{\alpha} \times k^{\beta} \to \mathrm{Aug}_m^a(T'; k) \times (k^*)^{\alpha'} \times k^{\beta'}$$

Notice also that, C_Z, C'_Z induce the same m-graded normal ruling of Z.

Conversely, given $(C_X, C_{Y'}, \underline{r'}, \underline{s'}, C'_Z)$, *define* $(C_Y, \underline{r}, \underline{s}) := \Phi_{h,Y}^{-1}(C_{Y'}, \underline{r'}, \underline{s'})$ and let $\varphi_R^{-1} : C_{Y'}|_{Y'_R} \xrightarrow{\sim} C_Y|_{Y_R}$ be the induced isomorphism in the proof of the theorem for Y, Y'. Then *define* $C_Z := \Phi'(\varphi_R^{-1}, C'_Z)$ as in Lemma 3.14. By a similar argument as above, we see that $(C_X, C_Y, \underline{r}, \underline{s}, C_Z)$ defines an element in $\mathrm{Aug}_m^a(T; k) \times (k^*)^{\alpha} \times k^{\beta}$. *Define* $\Phi'_h(C_X, C_{Y'}, \underline{r'}, \underline{s'}, C'_Z) := (C_X, C_Y, \underline{r}, \underline{s}, C_Z)$. Then we obtain an algebraic morphism

$$\Phi'_h : \mathrm{Aug}_m^a(T'; k) \times (k^*)^{\alpha'} \times k^{\beta'} \to \mathrm{Aug}_m^a(T; k) \times (k^*)^{\alpha} \times k^{\beta}$$

By Lemma 3.14, it's easy to see that Φ_h, Φ'_h are inverse to each other.

Thus, we obtain an algebraic isomorphism $\Phi_h : \mathrm{Aug}_m^a(T; k) \times (k^*)^{\alpha} \times k^{\beta} \xrightarrow{\sim} \mathrm{Aug}_m^a(T'; k) \times (k^*)^{\alpha'} \times k^{\beta'}$. By definition, Φ_h clearly commutes with $Id : \mathrm{Aug}_m^a(T_L; k) = \mathrm{Aug}_m^a(X_L; k) \xrightarrow{\sim} \mathrm{Aug}_m^a(T'_L; k) = \mathrm{Aug}_m^a(X_L; k)$, and is compatible with the ruling decomposition over $T_R = T'_R$. This finishes the proof of the theorem. □

By a more careful check, the proof of the previous theorem also shows the following:

Corollary 3.11. *In the setting of Theorem 3.10, given any m-graded normal rulings ρ_L, ρ_R of $T_L = T'_L, T_R = T'_R$ respectively, and any $\epsilon_L \in O_m(\rho_L; k)$, there's an isomorphism:*

$$\Phi_h : \mathrm{Aug}_m(T, \epsilon_L, \rho_R; k) \times (k^*)^{B(T')} \times k^{\dim' - B(T')} \xrightarrow{\sim} \mathrm{Aug}_m(T', \epsilon_L, \rho_R; k) \times (k^*)^{B(T)} \times k^{\dim - B(T)}$$

where $\dim = \dim \mathrm{Aug}_m(T, \epsilon_L, \rho_R; k), \dim' = \dim \mathrm{Aug}_m(T', \epsilon_L, \rho_R; k)$*, and $B(T), B(T')$ are the numbers of base points on T, T' respectively.*

In particular, the mixed Hodge polynomial of $\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; \mathbb{C})$*, up to a normalization, defines a 2-variable Legendrian isotopy invariant generalizing ruling polynomials:*

$$\begin{aligned} P_T^m(\rho_L, \rho_R; q, t) : &= H_c(\mathbb{C}^{\times}; x, y, t)^{-B(T)} H_c(\mathbb{C}; x, y, t)^{-\dim + B(T)} H_c(\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; \mathbb{C}); x, y, t) \\ &= (t + qt^2)^{-B(T)} (qt^2)^{-\dim + B(T)} H_c(\mathrm{Aug}_m(T, \epsilon_{\rho_L}, \rho_R; \mathbb{C}); x, y, t) \end{aligned}$$

where $q = xy$.

3.3. **An isomorphism lifting property.**

Lemma 3.12. *Let (T,μ) be an elementary Legendrian tangle: a single crossing, a left cusp, n parallel strands with a single base point, or a marked right cusp. Then there exists a natural algebraic action $B_m(T_L) \curvearrowright \mathrm{Aug}_m^a(T;k)$, which lifts the action $B_m(T_L) \curvearrowright \mathrm{Aug}_m^a(T_L;k)$ with respect to the left restriction map $r_L : \mathrm{Aug}_m^a(T;k) \to \mathrm{Aug}_m^a(T_L;k)$, and such that the ruling decomposition $\mathrm{Aug}_m^a(T;k) = \sqcup_{\rho\in\mathrm{NR}_T^m}\mathrm{Aug}_m^\rho(T;k)$ coincides with the stratification of $\mathrm{Aug}_m^a(T;k)$ induced by $B_m(T_L)$-orbits.*

Proof. The proof is a case-by-case argument.

Case 1: *If T is a single crossing q* connecting strands $k, k+1$. Use the identification between augmentations and A-form MCSs (Theorem 3.8), any element $C \in \mathrm{Aug}_m^a(T;k)$ is a m-graded (acyclic) A-form MCS, which can be written as $C = (\{(C_l,d_l)\}_{l=0}^2, \{x_l\}_{l=0}^2, H)$ (Definition 3.2). Here H consists of a single handleslide H_r between strands $k, k+1$ immediately before the crossing q, with coefficient $r \in k$ ($r = 0$ in the case when $|q| \neq 0(\mathrm{mod} m)$), and (C_l,d_l) is a $\mathbb{Z}/m$-graded filtered complex over $T|_{\{x=x_l\}}$ with $C_l = C(T|_{\{x=x_l\}})$, where $x_0 = x_L, x_2 = x_R$ are the x-coordinates of the left and right endpoints of T respectively, and x_1 is the x-coordinate of a vertical line between H_r and q. In particular, $C_0 = C(T_L), C_2 = C(T_R)$ (Definition 1.21). Denote $\mu_L = \mu|_{T|_{\{x=x_L\}}}, \mu_R = \mu|_{T|_{\{x=x_R\}}}$.

Take any group element $\phi_0 \in B_m(T_L)$, we want to construct an action of ϕ_0 on C, i.e. a m-graded A-form MCS $\phi_0 \cdot C$ of T, which we denote by $C' = (\{(C_l,d_l')_{l=0}^2\}, \{x_l\}_{l=0}^2, H')$ of T.

Define $(C_0,d_0') := \phi_0(C_0,d_0) = (C_0, \phi_0\circ d_0\circ\phi_0^{-1})$. Denote by $<\phi_0(e_i), e_j>$ the coefficient of e_j in $\phi_0(e_i)$. *Define* a handleslide $H_{r'}$ between strands $k, k+1$ immediately to the left of q, with coefficient r', where

$$r' := \frac{<\phi_0(e_k), e_{k+1}> + r <\phi_0(e_{k+1}), e_{k+1}>}{<\phi_0(e_k), e_k>}. \tag{3.3.1}$$

Claim: (C_0,d_0') and $H' := \{H_{r'}\}$ defines a m-graded (acyclic) A-form MCS C' on T via Lemma 3.5.

In fact, if $|q| = \mu_L(k) - \mu_L(k+1) \neq 0(\mathrm{mod} m)$, then $r = 0$ and $<\phi_0(e_k), e_{k+1}>= 0$. It follows that H_r' is m-graded. Moreover, notice that ϕ_0 is a m-graded filtered isomorphism and $d_0' = \phi_0 \cdot d_0$, then $<d_0'e_k, e_{k+1}>=<\phi_0^{-1}e_k, e_k><\phi_0\circ d_0(e_k), e_{k+1}>=<\phi_0^{-1}e_k, e_k><\phi_0 e_{k+1}, e_{k+1}><d_0e_k, e_{k+1}>= 0$ as $<d_0e_k, e_{k+1}>= 0$. Hence, by Lemma 3.5, the claim holds.

In this case, we can also construct a sequence of isomorphisms $\phi = \{\phi_l\}$ with $\phi_l : (C_l,d_l) \xrightarrow{\sim} (C_l,d_l')$ an isomorphism of $\mathbb{Z}/m$-graded filtered complexes, such that ϕ depends on (C,ϕ_0) algebraically.

In fact, *define* $\phi_1 := H_{r'}\circ\phi_0\circ H_r^{-1}$ and $\phi_2 := s_q\cdot\phi_1 = s_q\circ\phi_1\circ s_q^{-1}$, where s_q is the elementary transformation corresponding to the crossing q. We will show that $\phi := \{\phi_l\}_{l=0}^2 : C \xrightarrow{\sim} C'$ defines an isomorphism of m-graded A-form MCSs (Definition 3.7). In fact, the only nontrivial condition to check is that $\phi_2 : C_2 \xrightarrow{\sim} C_2$ is filtered. To check this condition, again the only nontrivial case is $<\phi_2(e_{k+1}), e_k>= 0$, whose proof is done by a direct calculation:

$$\begin{aligned} &<\phi_2(e_{k+1}), e_k>=<s_q\circ\phi_1(e_k), e_k>=<\phi_1(e_k), e_{k+1}> \\ = &<H_{r'}\circ\phi_0(e_k + re_{k+1}), e_{k+1}> \end{aligned}$$

$$\begin{aligned}
&= \ <\phi_0(e_k), e_k><H_{r'}(e_k), e_{k+1}> + <\phi_0(e_k), e_{k+1}><H_{r'}(e_{k+1}), e_{k+1}> \\
&+ \ r<\phi_0(e_{k+1}), e_{k+1}><H_{r'}(e_{k+1}), e_{k+1}> \\
&= \ -r'<\phi_0(e_k), e_k> + <\phi_0(e_k), e_{k+1}> + r<\phi_0(e_{k+1}), e_{k+1}> \\
&= \ 0
\end{aligned}$$

Notice that the calculation above also shows that the m-graded A-form MCS C' is uniquely characterized by the conditions that $(C_0, d'_0) = \phi_0(C_0, d_0)$ and ϕ defined above induces an isomorphism of m-graded A-form MCSs. It's then easy to see that the construction of $\phi_0 \cdot C = C'$ induces an algebraic action $B_m(T_L) \curvearrowright \mathrm{Aug}_m(T;k)$, which lifts the obvious action $B_m(T_L) \curvearrowright \mathrm{Aug}_m(T_L;k)$. Moreover, it also follows from the $\mathbb{Z}/m$-graded filtered isomorphisms ϕ_l's that the $B_m(T_L)$-action preserves the ruling decomposition $\mathrm{Aug}^a_m(T;k) = \sqcup_\rho \mathrm{Aug}^\rho_m(T;k)$.

Now, it suffices to show that, for any m-graded normal ruling ρ of T, $B_m(T_L)$ acts transitively on the stratum $\mathrm{Aug}^\rho_m(T;k)$. That is, for any $C = (\{(C_l, d_l)\}_{l=0}^2, \{x_l\}_{l=0}^2, H), C' = (\{(C_l, d'_l)\}_{l=0}^2, \{x_l\}_{l=0}^2, H')$ in $\mathrm{Aug}^\rho_m(T;k)$, with $H = \{H_r\}, H' = \{H_{r'}\}$ the handleslides immediately to the left of q, there exists a $\phi_0 \in B_m(T_L)$ such that $C' = \phi_0 \cdot C$. For simplicity, *denote* $\rho_L := \rho|_{T_L}$. As $B_m(T)$ acts transitively on $\mathrm{Aug}^{\rho_L}_m(T_L;k)$, it suffices to consider the case $(C_0, d_0) = (C_0, d'_0) = (C_0, d_{\rho_L})$, where d_{ρ_L} is the canonical differential on $C(T_L)$ associated to ρ_L (Remark 1.24).

If $|q| \neq 0 (\mathrm{mod}\, m)$. Then $r = r' = 0$. Hence, by Lemma 3.4 have $C' = \phi_0 \cdot C$ for $\phi_0 = Id \in B_m(T_L)$. From now on, we may also assume $|q| = 0 (\mathrm{mod}\, m)$.

If q is a m-graded departure of ρ. Then again $r = r' = 0$ and $\phi_0 = Id$ satisfy the desired requirement.

If q is a m-graded switch of ρ. Then $r, r' \neq 0$. Let $a = r'/r$ and s be the strand paired with $k+1$ via ρ. Clearly, $s \neq k, k+1$. *Define* $\phi_0 \in B_m(T)$ via $\phi_0(e_{k+1}) = a e_{k+1}, \phi_0(e_s) = a e_s$ and $\phi_0(e_p) = e_p$ for all $p \in I(T_L) \setminus \{k+1, s\}$. We claim that $C' = \phi_0 \cdot C$. In fact, clearly we have $\phi_0 \cdot d_{\rho_L} = d_{\rho_L}$, that is, $(C_0, d'_0) = \phi_0(C_0, d_0)$. Moreover, by definition, the coefficient r'' of the handleslide in $\phi_0 \cdot C$ is given by $r'' = \frac{<\phi_0(e_k), e_{k+1}> + r<\phi_0(e_{k+1}), e_{k+1}>}{<\phi_0(e_k), e_k>} = ar = r'$. Therefore, by Lemma 3.4, we have $\phi_0 \cdot C = C'$ as desired.

If q is a m-graded return of type $(R1)$ of ρ (see Figure 1.3 (bottom row, left)). Then $k \in U_{\rho_L}, k+1 \in L_{\rho_L}$, and $\rho_L^{-1}(k+1) < k < k+1 < \rho_L(k)$ with $\rho_L^{-1}(k+1) \in U_{\rho_L}, \rho_L(k) \in L_{\rho_L}$. Let $a := r' - r$. *Define* $\phi_0 \in B_m(T_L)$ via $\tilde{e}_k := \phi_0(e_k) := e_k + a e_{k+1}$ and $\tilde{e}_p := \phi_0(e_p) := e_p$ otherwise. Then $d_{\rho_L} \tilde{e}_p = e_{\rho_L(p)}$ for all $p \in I(T_L)$. It follows that $\phi_0^{-1} \cdot d_{\rho_L} = d_{\rho_L}$, or equivalently, $(C_0, d'_0) = \phi_0(C_0, d_0)$. Moreover, the coefficient of the handleslide in $\phi_0 \cdot C$ is $r'' = \frac{<\phi_0(e_k), e_{k+1}> + r<\phi_0(e_{k+1}), e_{k+1}>}{<\phi_0(e_k), e_k>} = a + r = r'$. As a consequence, by Lemma 3.4, we have $\phi_0 \cdot C = C'$ as desired.

If q is a m-graded return of type $(R2)$ (resp. $(R3)$)of ρ (see Figure 1.3 (bottom row, middle (resp. right))). Then $k, k+1 \in U_{\rho_L}$ (resp. $k, k+1 \in L_{\rho_L}$). Let i, j be the strands paired with $k, k+1$ via ρ respectively. Then $k < k+1 < i < j$ or $i < j < k < k+1$. Let $a := r' - r$. *Define* $\phi_0 \in B_m(T_L)$ via $\tilde{e}_k := \phi_0(e_k) := e_k + a e_{k+1}$, $\tilde{e}_i := \phi_0(e_i) := e_i + a e_j$, and $\tilde{e}_p := \phi_0(e_p) := e_p$ for all $p \in I(T_L) \setminus \{k, i\}$. We claim that $\phi_0 \cdot C = C'$. In fact, clearly we have $(\phi_0^{-1} \cdot d_{\rho_L})(\tilde{e}_p) = \tilde{e}_{\rho_L(p)}$ for all $p \in U_{\rho_L}$. That is, $\phi_0^{-1} \cdot d_{\rho_L} = d_{\rho_L}$, or equivalently, $(C_0, d'_0) = \phi_0(C_0, d_0)$. Moreover, the coefficient r'' of the handleslide in $\phi_0 \cdot C$ is $r'' = \frac{<\phi_0(e_k), e_{k+1}> + r<\phi_0(e_{k+1}), e_{k+1}>}{<\phi_0(e_k), e_k>} = a + r = r'$. As a consequence, by Lemma 3.4, we have $\phi_0 \cdot C = C'$ as desired.

Case 2: *If T is a single left cusp q* connecting strands $k, k+1$ of T_R. Use the identification between augmentations and A-form MCSs (Theorem 3.8), any element $C \in \mathrm{Aug}_m^a(T;k)$ can be written as $C = (\{(C_l, d_l)\}_{l=0}^1, \{x_l\}_{l=0}^1, H)$ (Definition 3.2). Here $H = \emptyset$ and (C_l, d_l) is a $\mathbb{Z}/m$-graded filtered complex over $T|_{\{x=x_l\}}$ with $C_l = C(T|_{\{x=x_l\}})$, where $x_0 = x_L, x_1 = x_R$ are the x-coordinates of the left and right endpoints of T respectively. Denote $\mu_L = \mu|_{T|_{\{x=x_L\}}}, \mu_R = \mu|_{T|_{\{x=x_R\}}}$.

Notice that $(C_1, d_1) = (C_0, d_0) \oplus \mathrm{Span}\{e_k, e_{k+1} : d_1 e_k = (-1)^{\mu_R(k)} e_{k+1}\}$ via Definition 3.2.(5d). It follows that the left restriction map $r_L : \mathrm{Aug}_m^a(T;k) \xrightarrow{\sim} \mathrm{Aug}_m^a(T_L;k)$ is an isomorphism. Hence, the lemma holds trivially.

Moreover, we remark that for any $C = (\{(C_l, d_l)\}_{l=0}^1, \{x_l\}_{l=0}^1, H)$ in $\mathrm{Aug}_m^a(T;k)$ with $H = \emptyset$, and any $\phi_0 \in B_m(T_L)$, denote $C' := \phi_0 \cdot C$, which can be written as $C' = (\{(C_l, d_l')\}_{l=0}^1, \{x_l\}_{l=0}^1, H')$ with $H' = \emptyset$. We can *define* a $\mathbb{Z}/m$-graded filtered isomorphism $\phi_1 : (C_1, d_1) \xrightarrow{\sim} (C_1, d_1')$ by $\phi_1|_{(C_0,d_0)} := \phi_0$ and $\phi_1|_{\mathrm{Span}\{e_k, e_{k+1} : d_1 e_k = (-1)^{\mu_R(k)} e_{k+1}\}} = Id$. Then $\phi := \{\phi_0, \phi_1\} : C \xrightarrow{\sim} C'$ defines an isomorphism of $\mathbb{Z}/m$-graded A-form MCSs (Definition 3.7), and depends algebraically on ϕ_0, C.

Case 3: *If T is n parallel strands with a single base point q on the strand k.* Use the identification between augmentations and A-form MCSs (Theorem 3.8), any element $C \in \mathrm{Aug}_m^a(T;k)$ can be written as $C = (\{(C_l, d_l)\}_{l=0}^1, \{x_l\}_{l=0}^1, H)$ (Definition 3.2). Here $H = \{c_r\}$ consists of a single labeled base point at q (Definition 3.1) with $r \in k^*$, and (C_l, d_l) is a $\mathbb{Z}/m$-graded filtered complex over $T|_{\{x=x_l\}}$ with $C_l = C(T|_{\{x=x_l\}})$, where $x_0 = x_L, x_1 = x_R$ are the x-coordinates of the left and right endpoints of T respectively. Denote $\mu_L = \mu|_{T|_{\{x=x_L\}}}, \mu_R = \mu|_{T|_{\{x=x_R\}}}$.

Notice that $(C_1, d_1) = c_r(C_0, d_0)$ via Definition 3.2.(5e), where c_r denotes the elementary transformation corresponding to the labeled base point c_r at q (see Section 3.1.3). Thus, there's a canonical isomorphism $\mathrm{Aug}_m^a(T;k) \xrightarrow{\sim} \mathrm{Aug}_m^a(T_L;k) \times k^*$ whose first factor is r_L and the second factor given by $C \to r$.

Take any group element $\phi_0 \in B_m(T_L)$, let's define the action $\phi_0 \cdot C$ of T, which we denote by $C' = (\{(C_l, d_l')_{l=0}^1\}, \{x_l\}_{l=0}^1, H')$ of T.

Define $(C_0, d_0') := \phi_0(C_0, d_0) = (C_0, \phi_0 \circ d_0 \circ \phi_0^{-1})$. Denote by $< \phi_0(e_i), e_j >$ the coefficient of e_j in $\phi_0(e_i)$. *Define* a labeled base point $c_{r'}$ at q, with r' given by

$$r' := \frac{r}{< \phi_0(e_k), e_k >}. \tag{3.3.2}$$

Clearly, (C_0, d_0') and $H' := \{c_{r'}\}$ defines a m-graded (acyclic) A-form MCS C' on T via Lemma 3.5.

In this case, we can also construct a sequence of isomorphisms $\phi = \{\phi_l\}$ with $\phi_l : (C_l, d_l) \xrightarrow{\sim} (C_l, d_l')$ an isomorphism of $\mathbb{Z}/m$-graded filtered complexes, such that ϕ depends on (C, ϕ_0) algebraically. In fact, *define* $\phi_1 := c_{r'} \circ \phi_0 \circ c_r^{-1}$. Then $\phi := \{\phi_l\}_{l=0}^2 : C \xrightarrow{\sim} C'$ defines an isomorphism of m-graded A-form MCSs (Definition 3.7). Moreover, $< \phi_1(e_k), e_k >= 1$. And, C' is uniquely determined by the conditions $(C_0, d_0') = \phi_0(C_0, d_0)$ and $< \phi_1(e_k), e_k >= 1$. It then follows that $\phi_0 \cdot C = C'$ induces an algebraic action $B_m(T_L) \curvearrowright \mathrm{Aug}_m(T;k)$, which lifts the obvious action $B_m(T_L) \curvearrowright \mathrm{Aug}_m(T_L;k)$, and preserves the ruling decomposition $\mathrm{Aug}_m^a(T;k) = \sqcup_\rho \mathrm{Aug}_m^\rho(T;k)$.

Now, it suffices to show that, for any m-graded normal ruling ρ of T, $B_m(T_L)$ acts transitively on the stratum $\mathrm{Aug}_m^\rho(T;k)$. That is, for any $C = (\{(C_l, d_l)\}_{l=0}^1, \{x_l\}_{l=0}^1, H)$, and $C' =$

$(\{(C_l, d'_l)\}_{l=0}^{1}, \{x_l\}_{l=0}^{1}, H')$ in $\mathrm{Aug}_m^{\rho}(T;k)$, with $H = \{c_r\}, H' = \{c_{r'}\}$ the labeled base points at q, there exists a $\phi_0 \in B_m(T_L)$ such that $C' = \phi_0 \cdot C$. In fact, as $B_m(T_L)$ acts transitively on $\mathrm{Aug}_m^{\rho_L}(T_L;k)$, it suffices to consider the case $(C_0, d_0) = (C_0, d'_0) = (C_0, d_{\rho_L})$. Now, let i be the strand paired with k via ρ_L, and denote $a := r/r' \in k^*$. *Define* $\phi_0 \in B_m(T_L)$ via $\phi_0(e_k) = ae_k, \phi_0(e_i) = ae_i$ and $\phi_0(e_p) = e_p$ otherwise. Then, clearly $\phi_0 \cdot d_0 = d'_0 = d_{\rho_L}$, and the coefficient of the labeled base point in $\phi_0 \cdot C$ is $r'' = \frac{r}{<\phi_0(e_k), e_k>} = r'$. Thus, by Lemma 3.4, we have $C' = \phi_0 \cdot C$ as desired.

Case 4: *If T is a marked right cusp q* connecting strands $k, k+1$ of T_L. Then any element C in $\mathrm{Aug}_m^{a}(T;k)$ may be written as $C = (\{(C_l, d_l)\}_{l=0}^{2}, \{x_l\}_{l=0}^{2}, H)$, where H consists of a single handleslide H_r between strands $k, k+1$ immediately before q, with coefficient $r \in k$ ($r = 0$ in the case when $m \neq 1$). Here $x_0 = x_L, x_2 = x_R$ are the x-coordinates of the left and right endpoints of T respectively, and x_1 is the x-coordinate of a vertical line between H_r and q. In particular, $C_0 = C(T_L), C_2 = C(T_R)$ (Definition 1.21). Denote $\mu_L = \mu|_{T|_{\{x=x_L\}}}, \mu_R = \mu|_{T|_{\{x=x_R\}}}$. Assume $n_L, n_R = n_L - 2$ are the numbers of the left and right endpoints of T respectively.

By definition, we know $(C_1, d_1) = H_r(C_0, d_0)$ and (C_2, d_2) is determined by the short exact sequence of $\mathbb{Z}/m$-graded filtered complexes:

$$0 \to \mathrm{Span}\{e_k, d_1 e_k\} \to (C_1, d_1) \xrightarrow{Q_k} (C_2, d_2) \to 0$$

where Q_k is the morphism defined as in Definition 3.2 (5c). *Define* $H' := \{H_{r'}\}$, with

$$r' := \frac{< \phi_0(e_k), e_{k+1} > + r < \phi_0(e_{k+1}), e_{k+1} >}{< \phi_0(e_k), e_k >}. \tag{3.3.3}$$

Then (C_0, d'_0) and H' *define* a m-graded A-form MCS $C' = (\{(C_l, d'_l)\}_{l=0}^{2}, \{x_l\}_{l=0}^{2}, H')$ via Lemma 3.5. *Define* $\phi_1 : (C_1, d_1) \xrightarrow{\sim} (C_1, d'_1)$ by $\phi_1 := H_{r'} \circ \phi_0 \circ H_r^{-1}$, which is clearly a $\mathbb{Z}/m$-graded filtered isomorphism. Similar to **Case 1**, C' is uniquely determined by the conditions $(C_0, d'_0) = \phi_0(C_0, d_0)$ and $< \phi_1(e_k), e_{k+1} >= 0$. This shows that the construction of $\phi_0 \cdot C$ induces an algebraic $B_m(T_L)$-action on $\mathrm{Aug}_m^{a}(T;k)$, which lifts the obvious action $B_m(T_L) \curvearrowright \mathrm{Aug}_m^{a}(T_L;k)$.

Now, there're 2 different ways to construct two $\mathbb{Z}/m$-graded filtered isomorphisms $\phi_2 : (C_2, d_2) \xrightarrow{\sim} (C_2, d'_2)$ (resp. $\phi'_2 : (C_2, d_2) \xrightarrow{\sim} (C_2, d'_2)$), with $\phi_2 = \phi_2[\phi_0, C]$ (resp. $\phi'_2 = \phi'_2[\phi_0, C]$) depends algebraically on ϕ_0, C. Unlike the previous cases, the sequence of isomorphisms $\phi := \{\phi_0, \phi_1, \phi_2\}$ (resp. $\phi' := \{\phi_0, \phi_1, \phi'_2\}$) is no longer an isomorphism of m-graded A-form MCSs. The construction is as follows.

Let $\tilde{e}_k := e_k, \tilde{e}_{k+1} := d_1 e_k$, and $\tilde{e}_p := e_p - \frac{<d_1 e_p, e_{k+1}>}{<d_1 e_k, e_{k+1}>} e_k$ for $p \neq k, k+1$. Then $\{\tilde{e}_p\}_{p=1}^{n_L}$ is a new basis for the $\mathbb{Z}/m$-graded filtered k-module C_1, and $< d_1 \tilde{e}_p, e_{k+1} >= 0$ for $p \neq k, k+1$. Clearly, for $j > k+1$, have $\tilde{e}_j = e_j$ and $d_1 \tilde{e}_j \in \mathrm{Span}\{\tilde{e}_p\}_{p>k+1}$. For $i < k$, the condition $< d_1 \tilde{e}_i, e_{k+1} >= 0$ implies that

$$\begin{aligned} d_1 \tilde{e}_i &= \sum_{i<l<k} < d_1 e_i, e_l > e_l + < d_1 e_i, e_k > e_k (\mathrm{mod}\ \mathrm{Span}\{e_p\}_{p>k+1}) \\ &= \sum_{i<l<k} < d_1 e_i, e_l > \tilde{e}_l + z_i e_k (\mathrm{mod}\ \mathrm{Span}\{e_p\}_{p>k+1}) \end{aligned}$$

for some $z_i \in k$. It follows that $0 =< d_1^2 \tilde{e}_i, e_{k+1} >= z_i < d_1 e_k, e_{k+1} >$, which then implies that $z_i = 0$. Hence, $d_1 \tilde{e}_i \in \mathrm{Span}\{\tilde{e}_p, p \neq k, k+1\}$ for $i \neq k, k+1$. As consequence, we obtain a direct sum decomposition of $\mathbb{Z}/m$-graded filtered complexes $(C_1, d_1) = (\mathrm{Span}\{\tilde{e}_p, p \neq$

$k, k+1\}, d_1) \oplus \mathrm{Span}\{\tilde{e}_k, d_1\tilde{e}_k = \tilde{e}_{k+1}\}$. *Define* an isomorphism $\psi_1 : (C_1, d_1) \xrightarrow{\sim} (C_1, d_1)$ by $\psi_1(\tilde{e}_p) := \tilde{e}_p$ for $p \neq k, k+1$, and $\psi_1(\tilde{e}_k) := \phi_1^{-1}(e_k), \psi_1(\tilde{e}_{k+1}) := \phi_1^{-1} \circ d_1(e_k)$. One can check directly that $\psi_1 \circ d_1 = d_1 \circ \psi_1$, so ψ_1 indeed defines an $\mathbb{Z}/m$-graded filtered isomorphism. *Define* $\tilde{\phi}_1 := \phi_1 \circ \psi_1 : (C_1, d_1) \xrightarrow{\sim} (C_1, d_1')$, then $\tilde{\phi}_1(e_k) = \tilde{e}_k = e_k$. We thus obtain an isomorphism of short exact sequences of $\mathbb{Z}/m$-graded filtered complexes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Span}\{e_k, d_1e_k\} \arrow[r] \arrow[d, "\cong"] & (C_1, d_1) \arrow[r, "Q_k"] \arrow[d, "\tilde{\phi}_1"] & (C_2, d_2) \arrow[r] \arrow[d, dashed, "\phi_2"] & 0 \\
0 \arrow[r] & \mathrm{Span}\{e_k, d_1'e_k\} \arrow[r] & (C_1, d_1') \arrow[r, "Q_k"] & (C_2, d_2') \arrow[r] & 0
\end{tikzcd}$$

In particular, we obtain an isomorphism $\phi_2 : (C_2, d_2) \xrightarrow{\sim} (C_2, d_2')$ of $\mathbb{Z}/m$-graded filtered complexes, which clearly depends algebraically on ϕ_0, C.

Similarly, define a new basis $\{\tilde{e}_p'\}$ for C_1 by $\tilde{e}_k' := e_k, \tilde{e}_{k+1}' := d_1'e_k$, and $\tilde{e}_p' := e_p - \frac{<d_1'e_p, e_{k+1}>}{<d_1'e_k, e_{k+1}>} e_k$ for $p \neq k, k+1$. By the same reason, we obtain a direct sum decomposition of $\mathbb{Z}/m$-graded filtered complexes $(C_1, d_1') = (\mathrm{Span}\{\tilde{e}_p', p \neq k, k+1\}, d_1') \oplus \mathrm{Span}\{\tilde{e}_k', d_1'\tilde{e}_k' = \tilde{e}_{k+1}'\}$. *Define* an isomorphism $\psi_1' : (C_1, d_1') \xrightarrow{\sim} (C_1, d_1')$ by $\psi_1'(\tilde{e}_p') := \tilde{e}_p'$ for $p \neq k, k+1$, and $\psi_1'(\tilde{e}_k') := \phi_1(e_k), \psi_1'(\tilde{e}_{k+1}') := \phi_1 \circ d_1(e_k)$. One can check directly that $\psi_1' \circ d_1' = d_1' \circ \psi_1'$, so ψ_1' indeed defines an $\mathbb{Z}/m$-graded filtered isomorphism. *Define* $\tilde{\phi}_1' := (\psi_1')^{-1} \circ \phi_1 : (C_1, d_1) \xrightarrow{\sim} (C_1, d_1')$, then $\tilde{\phi}_1'(e_k) = \tilde{e}_k = e_k$. We thus obtain an isomorphism of short exact sequences of $\mathbb{Z}/m$-graded filtered complexes:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Span}\{e_k, d_1e_k\} \arrow[r] \arrow[d, "\cong"] & (C_1, d_1) \arrow[r, "Q_k"] \arrow[d, "\tilde{\phi}_1'"] & (C_2, d_2) \arrow[r] \arrow[d, dashed, "\phi_2'"] & 0 \\
0 \arrow[r] & \mathrm{Span}\{e_k, d_1'e_k\} \arrow[r] & (C_1, d_1') \arrow[r, "Q_k"] & (C_2, d_2') \arrow[r] & 0
\end{tikzcd}$$

In particular, we obtain an isomorphism $\phi_2' : (C_2, d_2) \xrightarrow{\sim} (C_2, d_2')$ of $\mathbb{Z}/m$-graded filtered complexes, which depends algebraically on ϕ_0, C.

Altogether, we have shown that the $B_m(T_L)$-action on $\mathrm{Aug}_m^a(T; k)$ preserves the ruling decomposition. It suffices to show that, $B_m(T_L)$ acts transitively on each stratum. That is, for any m-graded normal ruling ρ, and any two elements $C = (\{(C_l, d_l)\}_{l=0}^2, \{x_l\}_{l=0}^2, H)$, and $C' = (\{(C_l, d_l')\}_{l=0}^2, \{x_l\}_{l=0}^2, H')$ in $\mathrm{Aug}_m^\rho(T; k)$, with $H = \{H_r\}, H' = \{H_{r'}\}$ the handleslides immediately to the left of q, there exists a $\phi_0 \in B_m(T_L)$ such that $C' = \phi_0 \cdot C$.

In fact, as $B_m(T_L)$ acts transitively on $\mathrm{Aug}_m^{\rho_L}(T_L; k)$, it suffices to consider the case $(C_0, d_0) = (C_0, d_0') = (C_0, d_{\rho_L})$. Now, let $a = r' - r$, and *define* $\phi_0 \in B_m(T_L)$ by $\tilde{e}_k := \phi_0(e_k) := e_k + ae_{k+1}$ and $\tilde{e}_p := \phi_0(e_p) := e_p$ otherwise. Then $d_{\rho_L}\tilde{e}_p = \tilde{e}_{\rho_L(p)}$ for all $p \in U_{\rho_L}$. It follows that $\phi_0 \cdot d_{\rho_L} = d_{\rho_L}$, or equivalently, $(C_0, d_0') = \phi_0 \cdot (C_0, d_0)$. Moreover, the coefficient of the handleslide in $\phi_0 \cdot C$ is $r'' = \frac{<\phi_0(e_k), e_{k+1}> + r <\phi_0(e_{k+1}), e_{k+1}>}{<\phi_0(e_k), e_k>} = r'$. Thus, by Lemma 3.4, we have $\phi_0 \cdot C = C'$ as desired.

Remark 3.13. By the constructions of ϕ_2, ϕ_2', we have $\phi_2'[\phi_0^{-1}, \phi_0 \cdot C] = (\phi_2[\phi_0, C])^{-1}$ and hence also $\phi_2[\phi_0^{-1}, \phi_0 \cdot C] = (\phi_2'[\phi_0, C])^{-1}$.

Now, we have finished the proof of the lemma. □

More generally, we can show the following *isomorphism lifting property*:

Lemma 3.14. *Let (T,μ) be a Legendrian tangle such that each right cusp is marked. Then there are two algebraic morphisms $\Phi : B_m(T_L;k)\times \mathrm{Aug}_m^a(T;k)\to \mathrm{Aug}_m^a(T;k)$ and $\Phi' : B_m(T_L;k)\times \mathrm{Aug}_m^a(T;k)\to \mathrm{Aug}_m^a(T;k)$ such that*

(1) Given any m-graded A-form MCS $C=(\{(C_l,d_l)\},\{x_l\},H)$ in $\mathrm{Aug}_m^a(T;k)$, and any $\phi_0\in B_m(T_L;k)$, denote $\Phi(\phi_0,C):=(\{(C_l,d'_l)\},\{x_l\},H')$ (resp. $\Phi'(\phi_0,C):=(\{(C_l,d''_l)\},\{x_l\},H'')$). Then $\phi_0:(C_0,d_0)\xrightarrow{\sim}(C_0,d'_0)$ (resp. $\phi_0:(C_0,d_0)\xrightarrow{\sim}(C_0,d''_0)$) is an isomorphism of $\mathbb{Z}/m$-graded filtered complexes.

(2) In the situation above, there's a canonical way to extend ϕ_0 to a sequence of isomorphisms $\phi=\{\phi_l\}$ (resp. $\phi'=\{\phi'_l\}$) with $\phi_l:(C_l,d_l)\xrightarrow{\sim}(C_l,d'_l)$ (resp. $\phi'_l:(C_l,d_l)\xrightarrow{\sim}(C_l,d''_l)$) an isomorphism of $\mathbb{Z}/m$-graded filtered complexes, such that $\phi=\phi[\phi_0,C]$ (resp. $\phi'=\phi'[\phi_0,C]$) depends algebraically on ϕ_0,C.

(3) $\Phi'(\phi_0^{-1},\Phi(\phi_0,C))=C$, and $\Phi(\phi_0^{-1},\Phi'(\phi_0,C))=C$.

(4) $\phi'_l[\phi_0^{-1},\Phi(\phi_0,C)]=(\phi_l[\phi_0,C])^{-1}$, and $\phi_l[\phi_0^{-1},\Phi'(\phi_0,C)]=(\phi'_l[\phi_0,C])^{-1}$.

Proof. Take any m-graded A-form MCS $C=(\{(C_l,d_l)\},\{x_l\},H)$ in $\mathrm{Aug}_m^a(T;k)$, and any $\phi_0\in B_m(T_L;k)$. By cutting the Legendrian tangle (T,μ) into elementary pieces, notice that we can construct the A-form MCS $\Phi(\phi_0,C),\Phi'(\phi_0,C)$, and the sequences of isomorphisms ϕ,ϕ' inductively. Hence, it suffices to show the lemma for the case when T is an elementary Legendrian tangle: a single crossing, a single left cusp, n parallel strands with a single base point, or a single marked right cusp.

Now, we only have to repeat the proof of Lemma 3.12: In **Case 1, 2, 3**, we simply *define* $\Phi(\phi_0,C)=\Phi'(\phi_0,C):=C'=\phi_0\cdot C$, and $\phi=\phi':=\{\phi_l\}$, where C',ϕ_l are constructed in each case as in the proof of Lemma 3.12. The result then follows from Lemma 3.12. In **Case 4**, we *define* $\Phi(\phi_0,C)=\Phi'(\phi_0,C):=C'=\phi_0\cdot C$, and $\phi:=\{\phi_0,\phi_1,\phi_2\},\phi':=\{\phi_0,\phi_1,\phi'_2\}$, where $\phi_0,\phi_1,\phi_2,\phi'_2$ are constructed as in **Case 4** in the proof of Lemma 3.12. Now, the result follows from Lemma 3.12 and Remark 3.13. □

Remark 3.15. We see from the proof that if (T,μ) is an elementary tangle: a single crossing, a left cusp, n-parallel strands, or a marked right cusp, then $\Phi=\Phi'$. However, they are not identical for a general Legendrian tangle (T,μ). Because the inductive construction of Φ,Φ' also involves the sequences of isomorphisms ϕ,ϕ' for each elementary piece of T, whose constructions are different when the elementary piece is a marked right cusp.

4. The combinatorics of the ruling decomposition

Let (T,μ) be any Legendrian tangle with base points so that each right cusp is marked. Fix the base field $k=\mathbb{C}$. Recall that, associated to the ruling decomposition

$$\mathrm{Aug}_m(T,\rho_L,\rho_R;k)=\sqcup_{\rho\in\mathrm{NR}_T^m(\rho_L,\rho_R)}\mathrm{Aug}_m^\rho(T,\rho_L,\rho_R;k)$$

for the augmentation variety $\mathrm{Aug}_m(T,\rho_L,\rho_R;k)$, there's a spectral sequence (Lemma 2.4) computing the mixed Hodge structure of the variety. This motivates the problem of understanding the gluing behavior of the pieces in the ruling decomposition. More generally, we will study the combinatorics of the *(full) ruling decomposition* (Definition 1.28,Theorem 1.29):

$$\mathrm{Aug}_m^a(T;k)=\sqcup_{\rho\in\mathrm{NR}_T^m}\mathrm{Aug}_m^\rho(T;k) \tag{4.0.4}$$

where $\mathrm{Aug}^a_m(T;k) := \{\epsilon \in \mathrm{Aug}_m(T;k) : \epsilon|_{T_L}$ is acyclic (Remark 1.24).$\}$ is the (full) augmentation variety of acyclic augmentations.

Similar to Definition 2.1, we define:

Definition 4.1. Given any stratified space $X = \sqcup_{\rho\in\mathcal{S}} X_\rho$ and any two indices $\rho', \rho \in \mathcal{S}$, we *say* $\rho' \leq^G \rho$, if $X_{\rho'} \subset \overline{X}_\rho$. We will call the partial order $\leq^G$ so defined the *geometric partial order* on $\mathcal{S}$.

It's expected that the full ruling decomposition is indeed a *stratification*. That is, *the frontier axiom* is satisfied: the closure of any stratum is a disjoint union of strata. We can then pursue an combinatorial description of the geometric partial order on the full ruling decomposition.

4.1. **Trivial Legendrian tangles.** We firstly deal with case when T is the trivial Legendrian tangle of n parallel strands. In this case, the description will be a direct generalization of the results in [Mel06, Rot09].

By Remark 1.24, we then have $X = \mathrm{Aug}_m(T;k) = \sqcup_{\rho\in\mathcal{S}} X_\rho$, where $X_\rho = \mathrm{Aug}^\rho_m(T;k) = \mathcal{O}_m(\rho;k)$, and $\mathcal{S} = \mathcal{S}^m_T$ is the set of all m-graded isomorphism types of T. In this case, X is a $B_m(T)$-variety (Definition 1.21), stratified by finitely many $B_m(T)$-orbits $\mathcal{O}_m(\rho;k)$. Hence, $X = \sqcup_\rho X_\rho$ satisfies the *frontier axiom* and is indeed a *Whitney stratification*. Similarly, so is the ruling decomposition for $\mathrm{Aug}^a_m(T;k)$.

Now, let's study the geometric partial order on $\mathcal{S}$. Given any m-graded isomorphism type ρ of T, recall that ρ is a partition $I = I(T) = U_\rho \sqcup L_\rho \sqcup H_\rho$ and a bijection $\rho : U_\rho \xrightarrow{\sim} L_\rho$ (Definition 1.23). Equivalently, we can write ρ as an involution in the symmetry group S_n of n letters. That is, $\rho = (i_1 j_1)(i_2 j_2)\ldots(i_k j_k) \in S_n$, if $U_\rho = \{i_1 < i_2 < \ldots < i_k\}$ and $L_\rho = \{j_1, j_2, \ldots, j_k\}$. Finally, it can also be identified with the canonical differential d_ρ on $C(T)$, given by $d_\rho e_i = e_{\rho(i)}$ for $i \in U_\rho$, and $d_\rho e_i = 0$ for $i \in L_\rho \sqcup H_\rho$.

Definition 4.2. Fix $1 \leq i \leq j \leq n$. We introduce some notations:

- *Define* $T_{i,j}$ to be the trivial Legendrian tangle of the strands $i, i+1, \ldots, j$ of T.
- For any d in $\mathcal{O}_m(\rho;k)$, that is, $(C(T), d)$ is a $\mathbb{Z}/m$-graded filtered complex whose Barannikov normal form (Lemma 1.22) is d_ρ, we *define* a differential $d_{i,j}$ on $C(T_{i,j})$ so that $(C(T_{i,j}), d_{i,j}) = (\mathrm{Span}\{e_l, l \geq i\}, d)/(\mathrm{Span}\{e_l, l > j\}, d)$ is the sub-quotient of $(C(T), d)$, as a $\mathbb{Z}/m$-graded filtered complex.
  In other words, $\pi_{i,j}(d)$ is represented by the sub-matrix $(de_p, e_q)_{i\leq p,q\leq j}$, consisting of the rows and columns $i, i+1, \ldots, j$ of $(< de_p, e_q >)_{1\leq p,q\leq n}$.
- *Define* $\rho_{i,j} := \rho|_{T_{i,j}}$ to be the m-graded isomorphism type on $T_{i,j}$ determined by $(d_\rho)_{i,j}$.
- It's clear that $d_{i,j} \in \mathcal{O}_m(T_{i,j};k)$. We then obtain an *induced map* $\pi_{i,j} : \mathcal{O}_m(\rho;k) \to \mathcal{O}_m(\rho_{i,j};k)$ by $\pi_{i,j}(d) := d_{i,j}$.
- For simplicity, for each $a = 0, 1, \ldots, m-1 (\mathrm{mod}\ m)$, and any subset J of $I(T)$, *define* $J^a := \{p \in J \| e_p| = \mu(p) = a (\mathrm{mod}\ m)\}$.

Definition 4.3. For any differential d in $\mathrm{Aug}_m(T;k)$, define a *rank matrix* $R(d) := (R_{p,q}(d))_{1\leq p,q\leq n}$, where $R_{p,q}(d) := (R^a_{p,q}(d))^{m-1}_{a=0}$ is a vector given by

$$R^a_{p,q}(d) := \begin{cases} 0 & p > q \\ \dim \mathrm{Span}\{\pi_{p,q}(d)e_k | p \leq k \leq q, |e_k| = a (\mathrm{mod}\ m)\} & p \leq q \end{cases}$$

That is, $R^a_{p,q}(d)$ is the rank of the matrix $(<\pi_{p,q}(d)e_k, e_l>)_{k\in I^a(T_{p,q}), l\in I^{a-1}(T_{p,q})}$ for $p \leq q$.

Notice also that, by the previous discussion, $R_{p,q}(d)$ is constant on the orbit $B_m(T)\cdot d = O_m(\rho;k)$, where ρ is the m-graded isomorphism type determined by d. So for each m-graded isomorphism type ρ of T, can define a *rank matrix* $R(\rho) := (R_{p,q}(\rho))_{1\leq p,q\leq n}$, by $R(\rho) := R(d_\rho)$. Equivalently, we have

$$R^a_{p,q}(\rho) = \#\{p \leq k \leq q \| |e_k| = a (\mathrm{mod}\ m), k \in U_\rho, \rho(k) \leq q\} \tag{4.1.1}$$

Convention: For simplicity, for any $\rho \in \mathcal{S}^m_T$ and any $i \leq j$, the rank matrix of $\rho_{i,j} \in \mathcal{S}^m_{T_{i,j}}$ will be denoted by $(R_{p,q}(\rho_{i,j}))_{i\leq p,q\leq j} = (R_{p,q}(\rho))_{i\leq p,q\leq j}$.

We now define an (algebraic) partial order on $\mathcal{S}^m_T$ as follows:

Definition 4.4. Let $\rho', \rho \in \mathcal{S}^m_T$ be any two m-graded isomorphism types of T, we say $\rho' \leq^A \rho$ or simply $\rho' \leq \rho$, if $R(\rho') \leq R(\rho)$, that is $R^a_{i,j}(\rho') \leq R^a_{i,j}(\rho)$ for all $1 \leq i \leq j \leq n$ and $a = 0, 1, \ldots, m-1 (\mathrm{mod}\ m)$. This partial order will be called the *algebraic partial order* on $\mathcal{S}^m_T$.

Proposition 4.5. *For any m-graded isomorphism type ρ of T, we have*

$$\overline{\mathrm{Aug}^\rho_m(T;k)} = \sqcup_{\rho'\in\mathcal{S}^m_T:\rho'\leq\rho} \mathrm{Aug}^{\rho'}_m(T;k) \tag{4.1.2}$$

where the closure is taken in $\mathrm{Aug}_m(T;k)$. *That is, $\rho' \leq^G \rho$ if and only if $\rho' \leq \rho$. Moreover, the Zariski closure $\overline{X}_\rho$ is set-theoretically defined by the conditions $R_{p,q}(d) \leq R_{p,q}(\rho)$ for all $1 \leq p < q \leq n$, in* $\mathrm{Aug}_m(T;k) = \{d | d : C(T) \to C(T)$ *is a filtration-preserving differential of degree* -1 *of the* $\mathbb{Z}/m$*-graded k-module* $C(T)\}$.

As an immediate consequence, we obtain

Corollary 4.6. *The ruling decomposition* $\mathrm{Aug}^a_m(T;k) = \sqcup_{\rho\in\mathrm{NR}^m_T} \mathrm{Aug}^\rho_m(T;k)$ *is a* Whitney stratification, *and for any m-graded normal ruling ρ of T, we have*

$$\overline{\mathrm{Aug}^\rho_m(T;k)} = \sqcup_{\rho'\in\mathrm{NR}^m_T:\rho'\leq\rho} \mathrm{Aug}^{\rho'}_m(T;k) \tag{4.1.3}$$

where the closure is taken in $\mathrm{Aug}^a_m(T;k)$.

To show the proposition, we need the following result. If $\rho = (i_1 j_1)\ldots(i_k j_k) \in \mathcal{S}^m_T$, *define* $l(\rho) := k$, and for each $1 \leq r \leq k$, we *say* $(i_r j_r) \in \rho$.

Lemma 4.7. *The algebraic partial order $\leq$ on $\mathcal{S} = \mathcal{S}^m_T$ is generated by the following relations:*

1. *Given any $\rho \in \mathcal{S}$ containing two 2-cycles (ij), $(i'j')$ (that is, $i, i' \in U_\rho$, $j, j' \in L_\rho$ and $\rho(i) = j, \rho(i') = j$) such that $i < j < i' < j'$ and $\mu(j) = \mu(i')(\mathrm{mod}\ m)$, $\rho_{j,i'} := \rho(ij)(i'j')(ii')(jj')$ defines a new element in $\mathcal{S}$. Then $\rho_{j,i'} \leq \rho$.*
2. *Given any $\rho \in \mathcal{S}$ containing two 2-cycles (ij), $(i'j')$ (that is, $i, i' \in U_\rho$, $j, j' \in L_\rho$ and $\rho(i) = j, \rho(i') = j'$) such that $i < i' < j' < j$ and $\mu(i) = \mu(i')(\mathrm{mod}\ m)$, $\rho_{i,i'} := \rho(ij)(i'j')(ij')(i'j)$ defines a new element in $\mathcal{S}$. Then $\rho_{i,i'} \leq \rho$.*
3. *Given any $\rho \in \mathcal{S}$ containing a 2-cycle (ij) and $h \in H_\rho$ such that $h < i$ and $\mu(h) = \mu(i)(\mathrm{mod}\ m)$, $\rho_{i\uparrow,h} := \rho(ij)(hj)$ defines a new element in $\mathcal{S}$. Then $\rho_{i\uparrow,h} \leq \rho$.*
4. *Given any $\rho \in \mathcal{S}$ containing a 2-cycle (ij) and $h \in H_\rho$ such that $h > j$ and $\mu(h) = \mu(j)(\mathrm{mod}\ m)$, $\rho_{j\downarrow,h} := \rho(ij)(ih)$ defines a new element in $\mathcal{S}$. Then $\rho_{j\downarrow,h} \leq \rho$.*

(5) *Given any $\rho \in \mathcal{S}$ containing a 2-cycle (ij), $\rho^-_{(ij)} := \rho(ij)$ defines a new element in $\mathcal{S}$. Then $\rho^-_{(ij)} \leq \rho$.*

As an immediate consequence, we obtain:

Corollary 4.8. *The partial order $\leq$ on NR_T^m is generated by the following relations:*

(1) *Given any $\rho \in \mathrm{NR}_T^m$ containing two 2-cycles (ij), $(i'j')$ (that is, $i, i' \in U_\rho$, $j, j' \in L_\rho$ and $\rho(i) = j, \rho(i') = j$) such that $i < j < i' < j'$ and $\mu(j) = \mu(i')(\mathrm{mod}\ m)$, $\rho_{j,i'} := \rho(ij)(i'j')(ii')(jj')$ defines a new element in NR_T^m. Then $\rho_{j,i'} \leq \rho$.*
(2) *Given any $\rho \in \mathrm{NR}_T^m$ containing two 2-cycles (ij), $(i'j')$ (that is, $i, i' \in U_\rho$, $j, j' \in L_\rho$ and $\rho(i) = j, \rho(i') = j'$) such that $i < i' < j' < j$ and $\mu(i) = \mu(i')(\mathrm{mod}\ m)$, $\rho_{i,i'} := \rho(ij)(i'j')(ij')(i'j)$ defines a new element in NR_T^m. Then $\rho_{i,i'} \leq \rho$.*

Assuming Lemma 4.7, we can show Proposition 4.5.

Proof of Proposition 4.5. One direction is easy. We have seen that $\overline{X}_\rho$ is a disjoint union of some $X_{\rho'}$'s. Notice also that the conditions $R_{p,q}(d) \leq R_{p,q}(\rho)$ can be expressed in terms of the vanishing of certain minors of $(< de_i, e_j >)_{1\leq i,j\leq n}$, so the conditions hold on X_ρ (which is automatic) implies that the same conditions also hold on $\overline{X}_\rho$. It follows that, $X_{\rho'} \subset \overline{X}_\rho$ implies $R_{p,q}(\rho') \leq R_{p,q}(\rho)$. In other words, $\rho' \leq^G \rho \Rightarrow \rho' \leq \rho$.

On the other hand, suppose $\rho' \leq \rho$, we want to show $X_{\rho'} \subset \overline{X}_\rho$. By Lemma 4.7, it suffices to consider the case when $\rho' = \rho_i$ is a m-graded isomorphism type obtained from ρ, as in Lemma 4.7.(i), for some $1 \leq i \leq 5$.

Case 1: Suppose $\rho' = \rho_{j,i'}$ is obtained from ρ as in Lemma 4.7.(1). That is, ρ contains two cycles $(ij), (i'j')$ such that $i < j < i' < j'$ and $\mu(j) = \mu(i')(\mathrm{mod}\ m)$, and $\rho' = \rho(ij)(i'j')(ii')(jj')$, viewed as an element of S_n. Consider the embedding of the affine line $d : k \to \mathrm{Aug}_m(T; k)$, given by $t \to d_t := d_{\rho'} + tE_{i,j} - tE_{i',j'}$. Here for all $1 \leq p \leq q \leq n$, we *define* $E_{p,q}$ to be the map $E_{p,q} : C(T) \to C(T)$, with $E_{p,q}(e_l) = 0$ for $l \neq p$, and $E_{p,q}(e_p) = e_q$. Recall that $d_{\rho'} : C(T) \to C(T)$ is the canonical differential determined by ρ'. It's then easy to see that $(C(T), d_t)$ is indeed a $\mathbb{Z}/m$-graded filtered complex, so the map d is well-defined.

For $t = 0$, $d_0 = d_{\rho'}$. For $t \neq 0$, d_t is $B_m(T)$-conjugate to d_ρ. In fact, define $\varphi_t \in B_m(T)$ as follows: Denote $\tilde{e}_p = \varphi_t^{-1}(e_p)$ for $1 \leq p \leq n$, then take $\tilde{e}_i := e_i, \tilde{e}_j := te_j + e_{i'}, \tilde{e}_{i'} := e_{i'}, \tilde{e}_{j'} := -te_{j'}$, and $\tilde{e}_p := e_p$ for $p \neq i, j, i', j'$. It follows that $d_t\tilde{e}_i = d_{\rho'}(e_i) + te_j = e_{i'} + te_j = \tilde{e}_j = \tilde{e}_{\rho(i)}$, $d_t\tilde{e}_{i'} = d_{\rho'}(e_{i'}) - te_{j'} = -te_{j'} = \tilde{e}_{j'} = \tilde{e}_{\rho(i')}$. Similarly, $d_t\tilde{e}_p = \tilde{e}_{\rho(p)}$ for $p \in U_\rho - \{i, i'\}$, and $d_t\tilde{e}_p = 0$ for all the remaining cases. In other words, $\varphi_t \cdot d_t = d_\rho$ as desired.

As a consequence, $d_t \in X_\rho = B_m(T) \cdot d_\rho \subset \overline{X}_\rho$ for $t \neq 0$. Hence, $d_0 = d_{\rho'} \in \overline{X}_\rho$. Since $\overline{X}_\rho$ is $B_m(T)$-invariant, we have $X_{\rho'} = B_m(T) \cdot d_{\rho'} \subset \overline{X}_\rho$.

Case 2: Suppose $\rho' = \rho_{i,i'}$ is obtained from ρ as in Lemma 4.7.(2). That is, ρ contains two cycles $(ij), (i'j')$, with $i < i' < j' < j$ and $\mu(i) = \mu(i')(\mathrm{mod}\ m)$, and $\rho' = \rho(ij)(i'j')(ij')(i'j)$, viewed as an element of S_n. Consider the embedding of the affine line $d : k \to \mathrm{Aug}_m(T; k)$, given by $t \to d_t := d_{\rho'} + tE_{i',j'}$. It's easy to check that d is well-defined.

For $t = 0$, $d_0 = d_{\rho'}$. For $t \neq 0$, d_t is $B_m(T)$-conjugate to d_ρ. In fact, define $\varphi_t \in B_m(T)$ as follows: Denote $\tilde{e}_p = \varphi_t^{-1}(e_p)$, then take $\tilde{e}_i := e_i - t^{-1}e_{i'}, \tilde{e}_{j'} := te_{j'} + e_j, \tilde{e}_j := -t^{-1}e_j$, and $\tilde{e}_p = e_p$ for $p \neq i, j', j$. It follows that $d_t(\tilde{e}_i) = d_{\rho'}(e_i - t^{-1}e_{i'}) - e_{j'} = -t^{-1}e_j = \tilde{e}_j = \tilde{e}_{\rho(i)}$,

$d_t(\tilde{e}_{i'}) = d_{\rho'}(e_{i'}) + te_{j'} = \tilde{e}_{j'} = \tilde{e}_{\rho(i')}$. Similarly, $d_t\tilde{e}_p = \tilde{e}_{\rho(p)}$ for $p \in U_\rho - \{i, i'\}$, and $d_t\tilde{e}_p = 0$ for all the remaining cases. In other words, $\varphi_t \cdot d_t = d_\rho$ as desired.

Now, as in Case 1, a similar argument shows that $X_{\rho'} \subset \overline{X}_\rho$.

Case 3: Suppose $\rho' = \rho_{i\uparrow,h}$ is obtained from ρ as in Lemma 4.7.(3). That is, ρ contains a cycle (ij) and $h \in H_\rho$, such that $h < i < j$ and $\mu(h) = \mu(i)(\text{mod } m)$, and $\rho' = \rho(ij)(hj)$, viewed as an element of S_n. Consider the embedding of the affine line $d : k \to \text{Aug}_m(T; k)$, given by $t \to d_t := d_{\rho'} + tE_{i,j}$. It's easy to see that d is well-defined.

For $t = 0$, $d_0 = d_{\rho'}$. For $t \neq 0$, d_t is $B_m(T)$-conjugate to d_ρ. In fact, define $\varphi_t \in B_m(T)$ as follows: Denote $\tilde{e}_p = \varphi_t^{-1}(e_p)$, then take $\tilde{e}_h := e_h - t^{-1}e_i, \tilde{e}_j := te_j$, and $\tilde{e}_p := e_p$ for $p \neq h, j$. It follows that $d_t\tilde{e}_h = 0, d_t\tilde{e}_i = d_{\rho'}(e_i) + te_j = \tilde{e}_j = \tilde{e}_{\rho(i)}$. Similarly, $d_t\tilde{e}_p = \tilde{e}_{\rho(p)}$ for $p \in U_\rho - \{i\}$, and $d_t\tilde{e}_p = 0$ for all the remaining cases. In other words, $\varphi_t \cdot d_t = d_\rho$ as desired.

Now, as in Case 1, a similar argument shows that $X_{\rho'} \subset \overline{X}_\rho$.

Case 4: Suppose $\rho' = \rho_{j\downarrow,h}$ is obtained from ρ as in Lemma 4.7.(4). That is, ρ contains a cycle (ij) and $h \in H_\rho$, such that $i < j < h$ and $\mu(j) = \mu(h)(\text{mod } m)$, and $\rho' = \rho(ij)(ih)$, viewed as an element of S_n. Consider the embedding of the affine line $d : k \to \text{Aug}_m(T; k)$, given by $t \to d_t := d_{\rho'} + tE_{i,j}$. It's easy to see that d is well-defined.

For $t = 0$, $d_0 = d_{\rho'}$. For $t \neq 0$, d_t is $B_m(T)$-conjugate to d_ρ. In fact, define $\varphi_t \in B_m(T)$ as follows: Denote $\tilde{e}_p = \varphi_t^{-1}(e_p)$, then take $\tilde{e}_j := te_j + e_h$, and $\tilde{e}_p := e_p$ for $p \neq j$. It follows that $d_t\tilde{e}_i = d_{\rho'}(e_i) + te_j = \tilde{e}_j = \tilde{e}_{\rho(i)}$. Similarly, $d_t\tilde{e}_p = \tilde{e}_{\rho(p)}$ for $p \in U_\rho - \{i\}$, and $d_t\tilde{e}_p = 0$ for all the remaining cases. In other words, $\varphi_t \cdot d_t = d_\rho$ as desired.

Now, as in Case 1, a similar argument shows that $X_{\rho'} \subset \overline{X}_\rho$.

Case 5: Suppose $\rho' = \rho^-_{(ij)}$ is obtained from ρ as in Lemma 4.7.(5). That is, ρ contains a cycle (ij) and $\rho' = \rho(ij)$, viewed as an element of S_n. Consider the embedding of the affine line $d : k \to \text{Aug}_m(T; k)$, given by $t \to d_t := d_{\rho'} + tE_{i,j}$. It's easy to see that d is well-defined.

For $t = 0$, $d_0 = d_{\rho'}$. For $t \neq 0$, d_t is clearly $B_m(T)$-conjugate to d_ρ. Thus, as in Case 1, a similar argument shows that $X_{\rho'} \subset \overline{X}_\rho$.

This finishes the proof of Proposition 4.5.

□

It suffices to show Lemma 4.7. For that purpose, we need some preparation. To modify the techniques in [Rot09] to our cases, we give another equivalent description of the algebraic partial order in Definition 4.4.

Fix the trivial Legendrian tangle (T, μ) of n parallel strands. As usual, label the strands from top to bottom by $1, 2, \ldots, n$.

Definition 4.9. A *m-graded word* (of length n) associated to T is a sequence of n letters $W = w_1 \ldots w_n$, with $w_i \in \{0, 1, \ldots, n\}$, such that $\mu(w_i) = \mu(i) + 1(\text{mod } m)$ if $w_i \neq 0$. Denote by $W_m(T)$ the set of all m-graded words associated to T.

Given a m-graded word $W = w_1 \ldots w_n$, define the $[i, j]$ *sub-word* to be the word $w_i \ldots w_j$.

Remark 4.10. Any m-graded isomorphism type ρ of T can be identified with a m-graded word $W_\rho = w_1 \ldots w_n$ of T, via $w_j := \rho^{-1}(j)$ for $j \in L_\rho$ and $w_j := 0$ otherwise. We will always use this identification.

Conversely, a m-graded word W is a m-graded isomorphism type if and only if: W has no repeated nonzero letter, and $w_j = i > 0$ implies that $i < j$ and $w_i = 0$.

Moreover, for any m-graded isomorphism type ρ, the rank matrix can alternatively be expressed in terms of the corresponding word $W_\rho = w_1 \dots w_n$. That is, for all $1 \leq p \leq q \leq n$ have

$$\begin{aligned} R^a_{p,q}(\rho) &= \#\{p \leq k \leq q | w_k \geq p, \mu(w_k) = a (\mathrm{mod}\, m)\} \\ &= \#(\{1 \leq k \leq q | w_k \geq p\}^{a-1}) \end{aligned} \tag{4.1.4}$$

where we have used the notation in Definition 4.2, and the fact that $w_k < k$ for all $1 \leq k \leq n$.

Definition 4.11. Given 2 m-graded words $V = v_1 \dots v_n$ and $W = w_1 \dots w_n$ of T, we say $V \leq W$ or W *covers* V, if for each $1 \leq l \leq n$, there exists a m-graded permutation σ_l of $\{1, 2, \dots, l\}$ (that is, $\mu(\sigma(k)) = \mu(k) (\mathrm{mod}\, m)$ for $1 \leq k \leq l$), such that $v_k \leq w_{\sigma_l(k)}$ for all $1 \leq i \leq l$. In this case, we say the set $\{v_1, \dots, v_l\}$ is covered by the set $\{w_1, \dots, w_l\}$ such that $w_{\sigma_l(k)}$ *covers* v_k. We call the sequence $\sigma_1, \dots, \sigma_n$ that demonstrates $V \leq W$ a *covering sequence*.

By definition, the set $W_m(T)$ is then equipped with a partial order.

Claim: For any 2 m-graded isomorphism types ρ', ρ of T, we have $\rho' \leq^A \rho$ if and only if W_ρ covers $W_{\rho'}$.

Proof of Claim. If $W_\rho = w_1 \dots w_n$ covers $W_{\rho'} = w'_1 \dots w'_n$, and $\sigma_1, \dots, \sigma_n$ is a covering sequence. In particular, $w_k < k, v_k < k$, and $w'_k \leq w_{\sigma_q(k)}$ for all $1 \leq k \leq q \leq n$. Then by Remark 4.10, for each $a = 0, 1, \dots, m-1 (\mathrm{mod}\, m)$ and $1 \leq p \leq q \leq n$, have

$$\begin{aligned} R^a_{p,q}(\rho) &= \#(\{1 \leq k \leq q | w_k \geq p\}^{a-1}) \\ &= \#(\{1 \leq k \leq q | w_{\sigma_q(k)} \geq p\}^{a-1}) \\ &\geq \#(\{1 \leq k \leq q | w'_k \geq p\}^{a-1}) \\ &= R^a_{p,q}(\rho') \end{aligned}$$

Hence, $\rho' \leq^A \rho$.

Conversely, if $\rho' \leq^A \rho$, by the same formula as above, for each fixed $a = 0, 1, \dots, m - 1 (\mathrm{mod}\, m)$, we see that $\#\{1 \leq k \leq q | w'_k \geq p\}^a \leq \#\{1 \leq k \leq q | w_k \geq p\}^a$ for all $1 \leq p \leq q \leq n$. Clearly, we can take m-graded permutations τ_q, τ'_q of $\{1, \dots, q\}$ such that $w_{\tau_q(i)} \leq w_{\tau_q(j)}$ (resp. $w'_{\tau'_q(i)} \leq w'_{\tau'_q(j)}$), whenever $i < j$ and $\mu(i) = \mu(j) (\mathrm{mod}\, m)$. It suffices to show that $w'_{\tau'_q(p)} \leq w_{\tau_q(p)}$ for all $1 \leq p \leq q \leq n$, as $\sigma_q := \tau_q \circ (\tau'_q)^{-1}$ will then form a covering sequence.

In fact, assume $\{k | 1 \leq k \leq q\}^a = \{k_1 < k_2 \dots < k_l\}$. For any $1 \leq i \leq l$, say $w'_{\tau'_q(k_i)} = p$ for some $0 \leq p \leq n$, we need to show $w'_{\tau'_q(k_i)} \leq w_{\tau_q(k_i)}$. The case $p = 0$ is trivial, so we can assume $1 \leq p \leq n$. Then $R^{a+1}_{p,q}(\rho') = \#\{1 \leq t \leq l | w'_{k_t} \geq p\} \geq l - i + 1$. It follows that $R^{a+1}_{p,q}(\rho) = \#\{1 \leq t \leq l | w_{k_t} \geq p\} \geq l - i + 1$, hence $w_{\tau_q(k_i)} \geq p = w'_{\tau'_q(k_i)}$. □

Suppose $V = v_1 \dots v_n, W = w_1 \dots w_n$ are two m-graded words with *no repeated nonzero letters* of T, such that $V \leq W$. There's a *standard covering sequence* that demonstrates $V \leq W$. The construction is defined in terms of a collection of subsets $\{A^a_i, B^a_i, C^a_i\}$ of $I(T)$ associated to V, W, for $0 \leq i \leq n$ and $a = 0, 1, \dots, m - 1 (\mathrm{mod}\, m)$:

Definition 4.12. Define $A_0^a = B_0^a = C_0^a := \emptyset$. $B_i^a := (\{v_1, v_2, \ldots, v_i\} - \{0\})^a$. Define A_i^a, C_i^a inductively, such that $(\{w_1, \ldots, w_i\} - \{0\})^a = A_i^a \sqcup C_i^a$ and $|A_i^a| = |B_i^a|$:

If $B_i^a = B_{i-1}^a$, i.e. $v_i = 0$ or $\mu(v_i) \neq a(\text{mod } m)$, then $A_i^a := A_{i-1}^a$ and $C_i^a := (C_{i-1}^a \cup \{w_i\} - \{0\})^a$.

Otherwise, $B_i^a = B_{i-1}^a \cup \{v_i\}$, i.e. $v_i \neq 0$ and $\mu(v_i) = a(\text{mod } m)$, so $\mu(w_i) = a(\text{mod } m)$ if $w_i \neq 0$. Denote $m_i^a := \max(C_{i-1}^a \cup \{w_i\} - \{0\})$, then $A_i^a := A_{i-1}^a \cup \{m_i^a\}$, and $C_i^a := C_{i-1}^a \cup \{w_i\} - \{0\} - \{m_i^a\}$.

Note: m_i^a exists. Otherwise, $w_i = 0$ and $C_{i-1}^a = \emptyset$, it follows that $|(\{v_1, \ldots, v_i\} - \{0\})^a| = |B_i^a| > |B_{i-1}^a| = |A_{i-1}^a| = |(\{w_1, \ldots, w_i\} - \{0\})^a| = |(\{w_{\tau_i(1)}, \ldots, w_{\tau_i(i)}\} - \{0\})^a|$, where τ_i is a m-graded permutation of $\{1, 2, \ldots, i\}$ such that $v_k \leq w_{\tau_i(k)}$ for all $1 \leq k \leq i$, coming from a covering sequence showing $V \leq W$. This is a contradiction.

As a direct generalization of [Rot09, Lem.4], we have:

Lemma 4.13. *Let $V \leq W$ be 2 m-graded words with no repeated nonzero letters of T, and let A_i^a, B_i^a, C_i^a be the subsets associated to V, W as in Definition 4.12. Suppose $|A_i^a| = |B_i^a| = j^a$. Then for all $1 \leq k \leq j^a$, the k-th largest letter of A_i^a is no less than the k-th largest letter of B_i^a. In particular, the sets A_i^a, B_i^a, C_i^a induce a covering showing $V \leq W$, with the k-th largest letter of A_i^a covers the largest k-th letter of B_i^a, the letters of C_i^a and the remaining zeroes of $\{w_1, \ldots, w_i\}$ cover zeroes of $\{v_1, \ldots, v_i\}$.*

Proof. The proof is similar to that in [Rot09, Lem.4]. Clearly, the second statement follows from the first. We show the first statement by induction on i. The case $i = 0$ is trivial. Assume the statement for $i - 1$, to show the result for i. Suppose that the elements of A_{i-1}^a, B_{i-1}^a are $a_1 > a_2 > \ldots > a_{j^a-1}$ and $b_1 > b_2 > \ldots > b_{j^a-1}$ respectively, so by induction, have $a_l \geq b_l$ for all $1 \leq l \leq j^a - 1$. If $B_i^a = B_{i-1}^a$, then $A_i^a = A_{i-1}^a$, the induction procedure is immediate. Now, suppose that a^*, b^* are the new letters added to A_{i-1}^a, B_{i-1}^a to obtain A_i^a, B_i^a respectively, then $a_k > a^* > a_{k+1}$ for some $0 \leq k \leq j^a - 1$. Recall that $a^* = \max(C_{i-1}^a \cup \{w_i\} - \{0\})$, so $a_1 > a_2 > \ldots > a_k > a^*$ are the $k + 1$ largest elements in $(\{w_1, \ldots, w_i\} - \{0\})^a$. As $V \leq W$, by definition we have $v_t \leq w_{\tau_i(t)}$ for all $1 \leq t \leq i$, for some m-graded permutation τ_i of $\{1, 2, \ldots, i\}$. It follows that $a_1 > a_2 > \ldots > a_k > a^*$ must cover (means "$\geq$") the $k + 1$ largest elements in $(\{v_1, \ldots, v_i\} - \{0\})^a = B_i^a$. Now, the remaining $j^a - k - 1$ elements $a_{k+1} > \ldots > a_{j^a-1}$ of A_i^a cover the $j^a - k - 1$ smallest elements $b_{k+1} > \ldots > b_{j^a-1}$ of B_{i-1}^a, which clearly cover the $j^a - k - 1$ smallest elements of B_i^a, as $B_{i_1}^a \subset B_i^a$. As a consequence, A_i^a covers B_i^a. □

Corollary 4.14. *Suppose $V = v_1 \ldots v_n \leq W = w_1 \ldots w_n$ are two m-graded words with no repeated nonzero letters of T, such that $|(\{w_i, \ldots, w_j\} - \{0\})^a| > |(\{v_i, \ldots, v_j\} - \{0\})^a|$. Then there exists a nonzero letter w_k with $i \leq k \leq j$, such that $w_k \in C_j^a$. In fact, $w_k \in C_m^a$ for all $k \leq m \leq j$, hence, there's a covering of the set $\{v_1, \ldots, v_m\}$ by the set $\{w_1, \ldots, w_m\}$ such that w_k covers a zero.*

Proof. By definition, $|C_j^a \setminus C_{i-1}^a| = |(\{w_i, \ldots, w_j\} - \{0\})^a| - |(\{v_i, \ldots, v_j\} - \{0\})^a| > 0$. So there exists some $w_k \in C_j^a \setminus C_{i-1}^a$. However, notice that $C_l^a \setminus C_{l-1}^a \subset (\{w_l\} - \{0\})^a$ for all $1 \leq l \leq n$, so $C_j^a \setminus C_{i-1}^a \subset (\{w_i, \ldots, w_j\} - \{0\})^a$. It follows that $i \leq k \leq j$ and $w_k \neq 0$. This shows the first statement. For the second statement, clearly $w_k \notin C_{k-1}^a$, so there exists some l with $k \leq l \leq j$, such that $w_k \in C_l^a, \ldots, C_j^a$ but $w_k \notin C_{l-1}^a$. But $C_l^a \setminus C_{l-1}^a \subset \{w_l\} - \{0\}$, it follows that $w_k = w_l$, hence $k = l$, that is $w_k \in C_k^a, \ldots, C_j^a$. Now, the second statement follows from the previous lemma. □

Finally, we're able to show Lemma 4.7.

Proof of Lemma 4.7. Clearly, for each of the five cases in the lemma, the corresponding partial order relation holds. It suffices to show that the algebraic partial order on $\mathcal{S}_T^m$ is generated by the five relations. That is, given $\rho \in \mathcal{S}_T^m$, for any $\rho' \in \mathcal{S}_T^m$ such that $\rho' < \rho$, then $\rho' \leq \rho_i$, where ρ_i is some m-graded isomorphism type in $\mathcal{S}_T^m$, constructed from ρ via one of the five cases in the lemma.

Denote by $W_\rho = w_1 \dots w_n, W_{\rho'} = w'_1 \dots w'_n$ the m-graded words of T corresponding to ρ, ρ' respectively.

Case 1: *If* $U_\rho^a \neq U_{\rho'}^a$. Notice that, $U_\rho^a = (\{w_1, \dots, w_n\} - \{0\})^a$ and $U_{\rho'}^a = (\{w'_1, \dots, w'_n\} - \{0\})^a$. Moreover, $W_{\rho'} < W_\rho$ implies that $|U_{\rho'}^a| \leq |U_\rho^a|$. It follows that U_ρ^a is not a subset of $U_{\rho'}^a$. Let $j = w_i$ be the *smallest element* of U_ρ^a such that, for some covering sequence showing $W_{\rho'} < W_\rho$, j always covers letters strictly less than itself.

Note: Any letter $w \in U_\rho^a - U_{\rho'}^a$ must cover letters strictly less than itself (if w covers some w', then $w' = 0$, or $w' \in U_{\rho'}^a$ and $w' \leq w$, hence $w' < w$ as $w \notin U_{\rho'}^a$). In particular, j is well-defined.

Now, for any $l < j$ such that $l = 0$ or $\mu(l) = a = \mu(j)(\text{mod } m)$, let $Z^l = z_1^l \dots z_n^l$ be the m-graded word defined by $z_i^l = l < j = w_i$ and $z_p^l = w_p$ for $p \neq i$. In general, Z^l is not a m-graded isomorphism type, but only an element in $W_m(T)$. By definition, we have $Z^l < W$, and $Z^k < Z^l$ in $W_m(T)$ for any $k < l < j$ such that $k = 0$ or $\mu(k) = \mu(l) = a(\text{mod } m)$.

Define s to be the maximal letter in $U_{\rho'}^a \cup \{0\}$ such that $s < j$. Then by definition of $j = w_i$, we have $W_{\rho'} \leq Z^s$. Now, *define* k to be the smallest letter in $\{p|0 < p < j\}^a \cup \{0\}$, such that $W_{\rho'} \leq Z^k < W_\rho$ in $W_m(T)$. Clearly, $k \leq s < j$, and by definition, this is equivalent to the following

Property: There's some covering sequence showing $W_{\rho'} < W_\rho$ such that $j = w_i$ covers no letter $> k$; And for every covering sequence, j must cover a letter $\geq k$ at some point.

Case 1.1: *If* $k = 0$ *or* $k \in H_\rho$. Then, Z^k defines a m-graded isomorphism type of T, with $Z^k = \rho^-_{(ji)}$ or $\rho_{j\uparrow,k}$, as in Lemma 4.7.(5) or (3). It follows that $W_{\rho'} \leq Z^k < W_\rho$, or equivalently, $\rho' \leq Z^k < \rho$, which shows Lemma 4.7. Similarly, for any $k < c < j$ such that $\mu(c) = a(\text{mod } m)$, if $c \in H_\rho^a$, then Z^c defines a m-graded isomorphism type of T, with $Z^c = \rho_{j\uparrow,c}$ as in Lemma 4.7.(3), and $W_{\rho'} \leq Z^c < W_\rho$, we're done.

From now on, we may assume $k > 0$ *and* $H_\rho^a \cap \{k, \dots, j-1\} = \emptyset$. Then $k \in U_\rho^a \cup L_\rho^a$.

Case 1.2: *If* $k \in U_\rho^a$. So $k = w_m < j$ for some $m \in L_\rho^{a-1}$. It follows that $k \in U_{\rho'}^a$, otherwise there exists some covering sequence showing $W_{\rho'} < W_\rho$ such that $k = w_m$ always covers letters less than itself, contradicting the fact that j is the minimal letter in U_ρ^a with this property.

If $m < i$. By the definition of k (see **Property**), there's some covering sequence $\sigma_1, \dots, \sigma_n$ showing $W_{\rho'} \leq W_\rho$, such that $j = w_i$ always covers letters $\leq k$. Moreover, for some $l \geq i$, $\{w_1, \dots, w_l\}$ covers $\{w'_1, \dots, w'_l\}$ via σ_l such that $j = w_i$ covers a letter k. As $m < i$, $k = w_m$ covers some $a \leq k$ in $\{w'_1, \dots, w'_l\}$ via σ_l. But k appears at most once in $W_{\rho'}$, $a < k$. Now, modify σ_l so that $k = w_m$ covers k and $j = w_i$ covers $a < k$, we then obtain a new covering sequence. Repeat this procedure, in the end we obtain a covering sequence where $j = w_i$ only covers letters $< k$, contradicting the definition of k (see **Property**).

Hence, $i < m$. Define a m-graded word $Z = z_1 \dots z_n$, by $z_i = k = w_m, z_m = j = w_i$ and $z_p = w_p$ for $p \neq i, m$. Clearly, Z defines a m-graded isomorphism type $Z = \rho_{k,j}$ as in Lemma 4.7.(2), so $Z < W_\rho$. Also, $Z^k < Z$ since $z^k_p = w_p = z_p$ for $p \neq i, m$, and $z^k_i = k = z_i, z^k_m = w_m = k < j = z_m$. Hence, $W_{\rho'} \leq Z^k < Z < W_\rho$, which implies that $\rho' < Z < \rho$, as desired.

Case 1.3: *If $k \in L^a_\rho$. If $k \in L_{\rho'}$ as well.* Then the word $W_{\rho'}$ does not contain the letter k. By the definition of k (**Property**), there's a covering sequence showing $W_{\rho'} < W_\rho$, such that $j = w_i$ covers letters $\leq k$, hence letters $< k$. This contradicts the definition of k (**Property**). Thus, *we must have $k \notin L_{\rho'}$*, equivalently, $w'_k = 0$.

Consider the $[k, j-1]$ sub-words of $W_\rho, W_{\rho'}$. Recall that, we may assume that $H^a_\rho \cap \{k, \dots, j-1\} = \emptyset$. For any $k \leq c \leq j-1$, if $c \in U^a_\rho$, then $c \in U^a_{\rho'}$. Otherwise, $c < j$ and $c \in U^a_\rho - U^a_{\rho'}$ contradicts the definition of j, as in any covering sequence showing $W_{\rho'} < W_\rho$, the letter c in W_ρ must cover letters less than itself in $W_{\rho'}$. Therefore, for any $k \leq c \leq j-1$, if $c \in L^a_{\rho'}$ (equivalently, $w'_c \neq 0$ and $\mu(c) = a(\text{mod } m)$), then $c \notin U^a_\rho$, which implies that $c \in L^a_\rho$ (equivalently, $w_c \neq 0$) by the previous assumption. Notice also that $w_k \neq 0, \mu(k) = a(\text{mod } m)$ but $w'_k = 0$. As a consequence, $|(\{w_k, \dots, w_{j-1}\} - \{0\})^{a+1}| > |(\{w'_k, \dots, w'_{j-1}\} - \{0\})^{a+1}|$. Since $k \in L^a_\rho$, the word Z^k has no repeated nonzero letters, and by definition of Z^k, $|(\{z^k_k, \dots, z^k_{j-1}\} - \{0\})^{a+1}| = |(\{w_k, \dots, w_{j-1}\} - \{0\})^{a+1}| > |(\{w'_k, \dots, w'_{j-1}\} - \{0\})^{a+1}|$. Notice also that $W_{\rho'} < Z^k$. Thus, by Corollary 4.14, there exists a nonzero letter $z^k_l = w_l$ in Z^k with $k \leq l \leq j-1$, such that $z^k_l \in C^{a+1}_{j-1}$. Here C^{a+1}_{j-1} is a subset of $I(T)$ associated to $W_{\rho'} < Z^k$ as in Definition 4.12.

Define a new m-graded word $Z^* = z^*_1 \dots z^*_n$ from Z^k, via $z^*_p = z^k_p$ for $p \neq l, j$, and $z^*_l = z^k_j (= w_j = 0)$, $z^*_j = z^k_l = w_l$. By construction of Z^*, it follows from Corollary 4.14 and $W_{\rho'} < Z^k$ that $W_{\rho'} \leq Z^* < Z^k < W_\rho$. Now, we construct a new m-graded word $Z = z_1 \dots z_n$ from W_ρ, via $z_p = w_p$ for $p \neq l, j, i$, and $z_l = 0 = w_j, z_j = w_l, z_i = l < j = w_i$. Clearly, $z_p \geq z^*_p$ for all $1 \leq p \leq n$, so $W_{\rho'} \leq Z^* \leq Z$. Notice also that Z defines a m-graded isomorphism type via $Z = \rho_{l,j}$, as in Lemma 4.7.(1). Hence, $\rho' \leq \rho_{l,j} < \rho$, as desired.

Apply Case 1 repeatedly, *we can assume from now on $U^a_\rho = U^a_{\rho'}$ for all $a = 0, \dots, m-1(\text{mod } m)$.*

Case 2: *If $L^a_\rho \neq L^a_{\rho'}$.* Notice that $|L^a_\rho| = |U^{a+1}_\rho| = |U^{a+1}_{\rho'}| = |L^a_{\rho'}|$, so there exists some $i \in L^a_{\rho'} - L^a_\rho$. Then $i \notin U^a_{\rho'} = U^a_\rho$. Thus, $i \in H^a_\rho$. It follows that $w'_i \neq 0$ and $w_i = 0$. As $W_{\rho'} < W_\rho$, we have $|(\{w_1, \dots, w_{i-1}\} - \{0\})^{a+1}| = |(\{w_1, \dots, w_i\} - \{0\})^{a+1}| \geq |(\{w'_1, \dots, w'_i\} - \{0\})^{a+1}| = |(\{w'_1, \dots, w'_{i-1}\} - \{0\})^{a+1}| + 1$. So by Corollary 4.14, there exists some nonzero letter $w_l \in C^{a+1}_{i-1}$ with $1 \leq l \leq i-1$. Here C^{a+1}_{i-1} is a subset of $I(T)$ associated to $W_{\rho'} < W_\rho$, as in Definition 4.12. Now, define a new m-graded word $Z = z_1 \dots z_n$ from W_ρ, via $z_p = w_p$ for $p \neq l, i$, and $z_l = 0 = w_i, z_i = w_l$. In other words, Z defines a m-graded isomorphism type $Z = \rho_{l\downarrow,i}$, as in Lemma 4.7.(4). In particular, $Z < W_\rho$. Moreover, by construction of Z, it follows from Corollary 4.14 and $W_{\rho'} < W_\rho$ that $W_{\rho'} \leq Z < W_\rho$, as desired.

Case 3: Apply Cases 1 and 2, *we can now assume $U^a_\rho = U^a_{\rho'}$ and $L^a_\rho = L^a_{\rho'}$ for all $a = 0, \dots, m-1(\text{mod } m)$.* It follows also that $H^a_\rho = H^a_{\rho'}$. In this case, the lemma follows from Claim 1 below.

□

Claim 1: Given $\rho' < \rho$ in the set $\mathcal{S}^m_T$ of m-graded isomorphism types on T, such that $U_{\rho'} = U_\rho, L_{\rho'} = L_\rho$, then $\rho' \leq \rho_i$, for some ρ_i constructed from ρ via Lemma 4.7.(1)-(4).

To show Claim 1, we need:

Claim 2: If $\rho', \rho \in \mathcal{S}_T^m$ satisfy that $\rho' < \rho$ and $R_{1,n}(\rho') \leq R_{1,n}(\rho) - 1_a$, then $\rho' \leq \rho_5 = \rho^-_{(ij)}$, for some 2-cycle (ij) contained in ρ such that $\mu(i) = a(\text{mod } m)$. Here 1_a is the vector with only one nonzero entry, which is 1 at the a-th slot, for some $a = 0, 1, \ldots, m-1(\text{mod } m)$.

Proof of Claim 2. We prove Claim 2 by induction on $n = \dim C(T)$, which is the number of strands in T. The claim is trivial if $\dim C(T) = 2$. Assume the claim holds for the case $\dim C(T) = n-1$, and consider the case $\dim C(T) = n$.

1. *If $n \notin L_\rho, L_{\rho'}$ (resp. $1 \notin U_\rho, U_{\rho'}$).* By symmetry, it suffices to consider the case $n \notin L_\rho, L_{\rho'}$, that is, $n \in H_\rho, H_{\rho'}$, then $R_{1,n-1}(\rho'_{1,n-1}) = R_{1,n}(\rho') \leq R_{1,n}(\rho) - 1_a = R_{1,n-1}(\rho_{1,n-1}) - 1_a$ and $\rho'_{1,n-1} < \rho_{1,n-1}$ in $\mathcal{S}^m_{T_{1,n-1}}$. By induction, there's a 2-cycle $(ij) \in \rho_{1,n-1}$ such that $\rho'_{1,n-1} \leq (\rho_{1,n-1})^-_{(ij)}$, and $\mu(i) = a(\text{mod } m)$. Then $\rho' \leq \rho_5 := \rho^-_{(ij)}$. In fact, for $q < n$, have $R_{p,q}(\rho') = R_{p,q}(\rho'_{1,n-1}) \leq R_{p,q}((\rho_{1,n-1})^-_{(ij)}) = R_{p,q}(\rho^-_{(ij)})$; It follows that $R_{p,n}(\rho') = R_{p,n-1}(\rho') \leq R_{p,n-1}(\rho) = R_{p,n}(\rho)$.

2. *If $n \notin L_{\rho'}$ and $n \in L_\rho$ (resp. $1 \notin U_{\rho'}$ and $1 \in U_\rho$).* Again, by symmetry, it suffices to consider the case $n \notin L_{\rho'}, n \in L_\rho$. Then $n \in H_{\rho'}$ and ρ contains a 2-cycle of the form (kn) for some $k < n$. Let $b := \mu(k)(\text{mod } m)$.

2.1. *If $b = a(\text{mod } m)$.* Then $\mu(k) = a(\text{mod } m)$. Moreover, $\rho' \leq \rho_5 := \rho^-_{(kn)}$. In fact, for $q < n$, have $R_{p,q}(\rho^-_{(kn)}) = R_{p,q}(\rho) \geq R_{p,q}(\rho')$; it follows that $R_{p,n}(\rho^-_{(kn)}) = R_{p,n-1}(\rho^-_{(kn)}) \geq R_{p,n-1}(\rho') = R_{p,n}(\rho')$ as $n \notin L_{\rho'}$.

2.2. *If $b \neq a(\text{mod } m)$.* Then $R_{1,n-1}(\rho_{1,n-1}) = R_{1,n-1}(\rho) \geq R_{1,n-1}(\rho') = R_{1,n-1}(\rho'_{1,n-1})$, and $R_{1,n-1}(\rho_{1,n-1}) = R_{1,n}(\rho) - 1_b \geq R_{1,n}(\rho') + 1_a - 1_b = R_{1,n-1}(\rho'_{1,n-1}) + 1_a - 1_b$ implies that $R^a_{1,n-1}(\rho_{1,n-1}) \geq R^a_{1,n-1}(\rho'_{1,n-1}) + 1$. Hence, $R_{1,n-1}(\rho'_{1,n-1}) \leq R_{1,n-1}(\rho_{1,n-1}) - 1_a$ and $\rho'_{1,n-1} < \rho_{1,n-1}$ in $\mathcal{S}^m_{T_{1,n-1}}$. By induction, $\rho_{1,n-1}$ contains a 2-cycle (ij) such that $\rho'_{1,n-1} \leq (\rho_{1,n-1})^-_{(ij)}$ and $\mu(i) = a(\text{mod } m)$. Then $\rho' \leq \rho_5 := \rho^-_{(ij)}$. In fact, for $q < n$, have $R_{p,q}(\rho') = R_{p,q}(\rho'_{1,n-1}) \leq R_{p,q}((\rho_{1,n-1})^-_{(ij)}) = R_{p,q}(\rho^-_{(ij)})$; It follows that $R_{p,n}(\rho') = R_{p,n-1}(\rho') \leq R_{p,n-1}(\rho^-_{(ij)}) \leq R_{p,n}(\rho^-_{(ij)})$.

3. *If $n \in L_{\rho'}$ and $1 \in U_{\rho'}$, and $(1n) \in \rho'$.* Let $b := \mu(1) = \mu(n) + 1(\text{mod } m)$. By the convention in Definition 4.3, recall that the rank matrix of $\rho_{2,n}$ (resp. $\rho'_{2,n}$) is denoted by $(R_{p,q}(\rho_{2,n}))_{2 \leq p,q \leq n}$ (resp. $(R_{p,q}(\rho'_{2,n}))_{2 \leq p,q \leq n}$). Then $R_{2,n}(\rho'_{2,n}) = R_{2,n}(\rho') = R_{1,n}(\rho') - 1_b \leq R_{1,n}(\rho) - 1_a - 1_b \leq R_{2,n}(\rho) - 1_a = R_{2,n}(\rho_{2,n}) - 1_a$ and $\rho'_{2,n} < \rho_{2,n}$ in $\mathcal{S}^m_{T_{2,n}}$. By induction, $\rho_{2,n}$ contains a 2-cycle (ij) such that $\rho'_{2,n} \leq (\rho_{2,n})^-_{(ij)}$ and $\mu(i) = a(\text{mod } m)$. Then $\rho' \leq \rho_5 := \rho^-_{(ij)}$. In fact, for $p > 1$, have $R_{p,q}(\rho^-_{(ij)}) = R_{p,q}((\rho_{2,n})^-_{(ij)}) \geq R_{p,q}(\rho'_{2,n}) = R_{p,q}(\rho')$; for $q < n$, have $R_{1,q}(\rho^-_{(ij)}) \geq R_{2,q}((\rho_{2,n})^-_{(ij)}) \geq R_{2,q}(\rho'_{2,n}) = R_{1,q}(\rho')$; $R_{1,n}(\rho^-_{(ij)}) = R_{1,n}(\rho) - 1_a \geq R_{1,n}(\rho')$.

4. *If $n \in L_{\rho'}$ and $1 \in U_{\rho'}$, and $(1n) \notin \rho'$.* Let $b := \mu(n) + 1(\text{mod } m)$.

4.1. *If $a \neq b(\text{mod } m)$.* Then $R_{1,n-1}(\rho'_{1,n-1}) = R_{1,n-1}(\rho') = R_{1,n}(\rho') - 1_b \leq R_{1,n}(\rho) - 1_a - 1_b \leq R_{1,n-1}(\rho_{1,n-1}) - 1_a$, and $\rho'_{1,n-1} < \rho_{1,n-1}$ in $\mathcal{S}^m_{T_{1,n-1}}$. By induction, $\rho_{1,n-1}$ contains a 2-cycle (ij) such that $\rho'_{1,n-1} \leq (\rho_{1,n-1})^-_{(ij)}$ and $\mu(i) = a(\text{mod } m)$. Then $\rho' \leq \rho_5 := \rho^-_{(ij)}$. In fact, for $q < n$, have $R_{p,q}(\rho') = R_{p,q}(\rho'_{1,n-1}) \leq R_{p,q}((\rho_{1,n-1})^-_{(ij)}) = R_{p,q}(\rho^-_{(ij)})$; It follows that, for $x \neq b(\text{mod } m)$, have $R^x_{p,n}(\rho') = R^x_{p,n-1}(\rho') \leq R^x_{p,n-1}(\rho^-_{(ij)}) \leq R^x_{p,n}(\rho^-_{(ij)})$; $R^b_{p,n}(\rho') \leq R^b_{p,n}(\rho) = R^b_{p,n}(\rho^-_{(ij)})$ as $\mu(i) = a \neq b(\text{mod } m)$.

4.2. *If $a = b(\text{mod } m)$ and $n \notin L_\rho$.* Then $R_{1,n-1}(\rho'_{1,n-1}) = R_{1,n-1}(\rho') = R_{1,n}(\rho') - 1_a \leq R_{1,n}(\rho) - 2 \cdot 1_a = R_{1,n-1}(\rho_{1,n-1}) - 2 \cdot 1_a$, and $\rho'_{1,n-1} < \rho_{1,n-1}$ in $\mathcal{S}^m_{T_{1,n-1}}$. We can apply the inductive hypothesis twice. Hence, by induction, $\rho_{1,n-1}$ contains two distinct 2-cycles $(ij),(i'j')$ such that $\rho'_{1,n-1} \leq ((\rho_{1,n-1})^-_{(ij)})^-_{(i'j')} < (\rho_{1,n-1})^-_{(ij)}$ and $\mu(i) = \mu(i') = a(\text{mod } m)$. Say $i < i'$, then $\rho' \leq \rho_5 := \rho^-_{(ij)}$. In fact, for $q < n$, have $R_{p,q}(\rho') = R_{p,q}(\rho'_{1,n-1}) \leq R_{p,q}(((\rho_{1,n-1})^-_{(ij)})^-_{(i'j')}) = R_{p,q}((\rho^-_{(ij)})^-_{(i'j')}) \leq R_{p,q}(\rho^-_{(ij)})$; It follows that $R_{p,n}(\rho') \leq R_{p,n-1}(\rho') + 1_a \leq R_{p,n-1}((\rho^-_{(ij)})^-_{(i'j')}) + 1_a = R_{p,n-1}(\rho^-_{(ij)})$ if $p \leq i'$; If $p > i' > i$, have $R_{p,n}(\rho') \leq R_{p,n}(\rho) = R_{p,n}(\rho^-_{(ij)})$.

4.3. *If $a = b(\text{mod } m)$ and $n \in L_\rho$.* Then $R_{1,n-1}(\rho'_{1,n-1}) = R_{1,n-1}(\rho') = R_{1,n}(\rho') - 1_a \leq R_{1,n}(\rho) - 2 \cdot 1_a = R_{1,n-1}(\rho_{1,n-1}) - 1_a$, and $\rho'_{1,n-1} < \rho_{1,n-1}$ in $\mathcal{S}^m_{T_{1,n-1}}$. By induction, $\rho_{1,n-1}$ contains a 2-cycle (ij) such that $\rho'_{1,n-1} \leq (\rho_{1,n-1})^-_{(ij)}$ and $\mu(i) = a(\text{mod } m)$. Then $\rho' \leq \rho_5 := \rho^-_{(ij)}$. In fact, for $q < n$, have $R_{p,q}(\rho') = R_{p,q}(\rho'_{1,n-1}) \leq R_{p,q}((\rho_{1,n-1})^-_{(ij)}) = R_{p,q}(\rho^-_{(ij)})$; It follows that $R_{p,n}(\rho') \leq R_{p,n-1}(\rho') + 1_a \leq R_{p,n-1}(\rho^-_{(ij)}) + 1_a = R_{p,n}(\rho^-_{(ij)})$.

This finishes the proof of Claim 2. □

Now, we can prove Claim 1.

Proof of Claim 1. We prove the claim by induction on $n = \dim C(T)$. The first nontrivial case is $n = 4$, when $\rho' = (13)(24) < \rho = (12)(34)$ or $\rho' = (13)(24) < \rho = (14)(23)$. In the first situation, we have $\rho' = \rho_1 := \rho_{2,3}$, where $\rho_{2,3}$ is defined as in Lemma 4.7.(1); In the second situation, we have $\rho' = \rho_2 := \rho_{1,2}$, where $\rho_{1,2}$ is defined as in Lemma 4.7.(2). Hence the claim holds. Assume the Lemma holds for $\dim C(T) = n - 1$. Consider the case $\dim C(T) = n$.

1.*If $n \notin L_\rho$.* Then $n \in H_\rho, H_{\rho'}$, we can identify ρ, ρ' with $\rho_{1,n-1}, \rho'_{1,n-1}$ via the inclusion $S_{n-1} \hookrightarrow S_n$ of symmetry groups. Now, we have, $U_{\rho'_{1,n-1}} = U_{\rho_{1,n-1}}, L_{\rho'_{1,n-1}} = L_{\rho_{1,n-1}}$, and $\rho'_{1,n-1} < \rho_{1,n-1}$ in $\mathcal{S}^m_{T_{1,n-1}}$. By induction, $\rho'_{1,n-1} \leq (\rho'_{1,n-1})_i$ for some $1 \leq i \leq 4$ and some $(\rho'_{1,n-1})_i$ constructed from $\rho_{1,n-1}$ via case (i) in Lemma 4.7. It follows that $\rho' \leq \rho_i$, where ρ_i is identified with $(\rho_{1,n-1})_i$ via the inclusion $S_{n-1} \hookrightarrow S_n$, hence is constructed from ρ via Lemma 4.7.(i).

2. *If $(kn) \in \rho$ and $(kn) \notin \rho'$ for some $k < n$.* Let $a = \mu(k) = \mu(n) + 1(\text{mod } m)$. Since $U_\rho = U_{\rho'}, L_\rho = L_{\rho'}$, we have $k \in U^a_{\rho'}$. Then $k \notin U_{\rho_{1,n-1}}$ and $k \in U_{\rho'_{1,n-1}}$. It follows that $R_{k+1,n-1}(\rho'_{k+1,n-1}) = R_{k+1,n-1}(\rho') = R_{k,n-1}(\rho') - 1_a \leq R_{k,n-1}(\rho) - 1_a = R_{k+1,n-1}(\rho_{k+1,n-1}) - 1_a$, and $\rho'_{k+1,n-1} < \rho_{k+1,n-1}$ in $\mathcal{S}^m_{T_{k+1,n-1}}$. By Claim 1, $\rho_{k+1,n-1}$ contains a 2-cycle (ij) such that $\rho'_{k+1,n-1} \leq (\rho_{k+1,n-1})^-_{(ij)}$ and $\mu(i) = a(\text{mod } m)$. As $(kn),(ij) \in \rho, k < i < j < n$ and $\mu(k) = \mu(i) = a(\text{mod } m)$, $\rho_2 := \rho_{k,i}$ is well-defined as in Lemma 4.7.(2). Then $\rho' \leq \rho_2 = \rho_{k,i}$. In fact, for $p \geq k + 1$ and $q \leq n - 1$, have $R_{p,q}(\rho') = R_{p,q}(\rho'_{k+1,n-1}) \leq R_{p,q}((\rho_{k+1,n-1})^-_{(ij)}) = R_{p,q}(\rho^-_{(ij)}) = R_{p,q}(\rho_{k,i})$; for $p \leq k$ or $q = n$, have $R_{p,q}(\rho') \leq R_{p,q}(\rho) = R_{p,q}(\rho_{k,i})$.

3. *If $(kn) \in \rho, \rho'$ for some $k < n$.* Take $\tilde{T}$ to be the trivial Legendrian tangle obtained from T by removing strands k, n. Then the restrictions of ρ', ρ on $\tilde{T}$ define two m-graded isomorphism types on $\tilde{T}$, denoted by $\tilde{\rho}', \tilde{\rho}$ respectively. Clearly, $\tilde{\rho}' < \tilde{\rho}$ in $\mathcal{S}^m_{\tilde{T}}$, and $U_{\tilde{\rho}} = U_{\tilde{\rho}'}, L_{\tilde{\rho}} = L_{\tilde{\rho}'}$. By induction, $\tilde{\rho}' \leq \tilde{\rho}_i < \tilde{\rho}$ for some $1 \leq i \leq 4$ and some $\tilde{\rho}_i$ constructed from $\tilde{\rho}$, as in Lemma 4.7.(i) for $\tilde{T}$. Notice that, the construction of $\tilde{\rho}_i$ doesn't involve the strands k, n of T, so the same procedure constructs a m-graded isomorphism type ρ_i of T from ρ, as in Lemma 4.7.(i) for T. In other words, via the natural embedding of symmetry groups $S(I(\tilde{T})) \hookrightarrow S(I(T)) = S_n$, we have $\rho_i = \tilde{\rho}_i(kn)$. Similarly, $\rho = \tilde{\rho}(kn), \rho' = \tilde{\rho}'(kn)$. It follows that $\rho' \leq \rho_i < \rho$, as desired.

□

4.2. **Elementary Legendrian tangles.** By almost the same strategy, we then deal with the case when T is an elementary Legendrian tangle: a single crossing, a single left cusp, a single marked right cusp, or n-parallel strands with a single base point. For simplicity, for any m-graded normal ruling ρ of T, we *denote* $\rho_L := \rho|_{T_L}, \rho_R := \rho|_{T_R}$.

Lemma 4.15. *Let (T,μ) be an elementary Legendrian tangle: a single crossing, a left cusp, n parallel strands with a single base point, or a marked right cusp. Then the ruling decomposition* $\mathrm{Aug}_m^a(T;k) = \sqcup_{\rho\in\mathrm{NR}_T^m}\mathrm{Aug}_m^\rho(T;k)$ *is a* Whitney stratification*, and for any m-graded normal ruling ρ of T, we have:*

$$\overline{\mathrm{Aug}_m^\rho(T;k)} = \sqcup_{\rho'\in\mathrm{NR}_T^m:\rho'_L\leq\rho_L,\rho'_R\leq\rho_R}\mathrm{Aug}_m^{\rho'}(T;k) \tag{4.2.1}$$

where the closure is taken in $\mathrm{Aug}_m^a(T;k)$. *That is, for any $\rho,\rho'\in\mathrm{NR}_T^m$, $\rho'\leq^G\rho$ if and only if $\rho'_L\leq\rho_L,\rho'_R\leq\rho_R$.*

Proof. Recall that, by Lemma 3.12, the ruling decomposition is the same as the stratification by $B_m(T_L;k)$-orbits. In particular, it's indeed a *Whitney stratification*.

Now, let's describe the geometric partial order of this ruling stratification. Clearly, if $\rho'\leq^G\rho$, then $\rho'_L\leq\rho_L,\rho'_R\leq\rho_R$. It suffices to show the converse, i.e. if $\rho'_L\leq\rho_L,\rho'_R\leq\rho_R$. We use the notations in the proof of Lemma 3.12, and prove the result case by case.

If T is a single crossing q connecting strands $k,k+1$. Recall that any element C in $\mathrm{Aug}_m^a(T;k)$ is of the form $C = (\{(C_l,d_l)_{l=0}^2\},\{x_l\}_{l=0}^2, H=\{H_r\})$ as in **Case 1** of the proof of Lemma 3.12. Here H_r is the single handleslie immediately to the left of q such that $r=0$ if $|q|\neq 0(\mathrm{mod}\ m)$. Moreover, $(C_0,d_0),(C_1,d_1)=H_r(C_0,d_0),(C_2,d_2)=s_q(C_1,d_1)$ are all $\mathbb{Z}/m$-graded filtered complexes, equivalently, $<d_0e_k,e_{k+1}>=0$ by Lemma 3.5.

If $|q|\neq 0(\mathrm{mod}\ m)$. Then $r_L:\mathrm{Aug}_m^a(T;k)\xrightarrow{\sim}\mathrm{Aug}_m^a(T_L;k)$ is an isomorphism preserving the Ruling stratification. Thus, $\rho'\leq^G\rho$ follows from Corollary 4.6. From now on, we may assume $|q|=0(\mathrm{mod}\ m)$.

Consider the restrictions $r_L:\mathrm{Aug}_m^a(T;k)\to\mathrm{Aug}_m^a(T_L;k)$ and $r_R:\mathrm{Aug}_m^a(T;k)\to\mathrm{Aug}_m^a(T_R;k)$. Recall that $r_L(C)=(C_0,d_0)$, $r_R(C)=(C_2,d_2)$, for all C in $\mathrm{Aug}_m^a(T;k)$. By **Case 1** in the proof of Lemma 3.12, r_L,r_R preserve the Ruling stratifications. We thus obtain two decompositions

$$\mathrm{Aug}_m^a(T;k) = \sqcup_{\rho_0}r_L^{-1}(\mathrm{Aug}_m^{\rho_0}(T_L;k)) = \sqcup_{\rho_2}r_R^{-1}(\mathrm{Aug}_m^{\rho_2}(T_R;k)) \tag{4.2.2}$$

where ρ_0 (resp. ρ_2) runs over all $\rho_0\in\mathrm{NR}_{T_L}^m$ (resp. $\rho_2\in\mathrm{NR}_{T_R}^m$) such that ρ_0 (resp. ρ_2) doesn't pair the strands $k,k+1$, i.e. $(kk+1)\notin\rho_0$ (resp. $(kk+1)\notin\rho_2$).

Observe that, any element C in $r_L^{-1}(\mathrm{Aug}_m^{\rho_0}(T_L;k))$ is represented by $(C_0,d_0),H_r$ such that $d_0\in\mathrm{Aug}_m^{\rho_0}(T_L;k)$ (Note: this automatically implies that $<d_0e_k,e_{k+1}>=0$). Hence, $r_L^{-1}(\mathrm{Aug}_m^{\rho_0}(T_L;k))\cong\mathrm{Aug}_m^{\rho_0}(T_L;k)\times k$ via the map $C\to((C_0,d_0),r)$. Also, $\mathrm{Aug}_m^a(T;k)\cong\{(C_0,d_0)\in\mathrm{Aug}_m^a(T_L;k)|<d_0e_k,e_{k+1}>=0\}\times k$ via the same map. It follows that

$$\overline{r_L^{-1}(\mathrm{Aug}_m^{\rho_0}(T_L;k))} = \sqcup_{\rho'_0\leq\rho_0}r_L^{-1}(\mathrm{Aug}_m^{\rho'_0}(T_L;k)) \tag{4.2.3}$$

Similarly, any element C in $r_R^{-1}(\mathrm{Aug}_m^{\rho_2}(T_R;k))$ can be represented by $(C_2,d_2),H_r$ such that $d_2\in\mathrm{Aug}_m^{\rho_2}(T_L;k)$. Because the condition automatically gives $<d_2e_k,e_{k+1}>=0$, which then implies that $(C_1,d_1):=s_q^{-1}(C_2,d_2),(C_0,d_0):=H_r^{-1}(C_1,d_1)$ are $\mathbb{Z}/m$-graded filtered complexes.

Hence, $r_R^{-1}(\mathrm{Aug}_m^{\rho_2}(T_R;k)) \cong \mathrm{Aug}_m^{\rho_2}(T_R;k)\times k$ via the map $C \to ((C_2,d_2),r)$. Also, $\mathrm{Aug}_m^{a}(T;k) \cong \{(C_2,d_2)\in \mathrm{Aug}_m^{a}(T_R;k) | < d_2e_k, e_{k+1} >= 0\}\times k$ via the same map. It follows that

$$\overline{r_R^{-1}(\mathrm{Aug}_m^{\rho_2}(T_2;k))} = \sqcup_{\rho_2'\leq\rho_2} r_R^{-1}(\mathrm{Aug}_m^{\rho_2'}(T_R;k)) \tag{4.2.4}$$

In particular, the above two decompositions are stratifications.

Now, we come back to prove $\rho'\leq\rho$, given $\rho_L'\leq\rho_L, \rho_R'\leq\rho_R$. *If q is a m-graded return*, then $r_L^{-1}(\mathrm{Aug}_m^{\rho_L}(T_L;k)) = \mathrm{Aug}_m^{\rho}(T;k)$. *If q is a switch*, then $r_L^{-1}(\mathrm{Aug}_m^{\rho_L}(T_L;k)) = \mathrm{Aug}_m^{\rho}(T;k)\sqcup \mathrm{Aug}_m^{\tilde{\rho}}(T;k)$, where $\tilde{\rho}$ is the unique m-graded normal ruling of T such that, $\tilde{\rho}|_{T_L}=\rho_L$ and q is a m-graded departure. Then $\mathrm{Aug}_m^{\rho}(T;k)$ is open dense in $r_L^{-1}(\mathrm{Aug}_m^{\rho_L}(T_L;k))$. In both cases, we have

$$\begin{aligned}\overline{\mathrm{Aug}_m^{\rho}(T;k)} &= \overline{r_L^{-1}(\mathrm{Aug}_m^{\rho_L}(T_L;k))}\\ &= \sqcup_{\rho_0'\leq\rho_L} r_L^{-1}(\mathrm{Aug}_m^{\rho_0'}(T_L;k)) \supset r_L^{-1}(\mathrm{Aug}_m^{\rho_L'}(T_L;k)) \supset \mathrm{Aug}_m^{\rho'}(T;k)\end{aligned}$$

That is, $\rho'\leq^G\rho$. It suffices to consider the case when *q is a m-graded departure of ρ.* Then we have $r_R^{-1}(\mathrm{Aug}_m^{\rho_R}(T_R;k)) = \mathrm{Aug}_m^{\rho}(T;k)$. It follows that

$$\begin{aligned}\overline{\mathrm{Aug}_m^{\rho}(T;k)} &= \overline{r_R^{-1}(\mathrm{Aug}_m^{\rho_R}(T_R;k))}\\ &= \sqcup_{\rho_2'\leq\rho_R} r_R^{-1}(\mathrm{Aug}_m^{\rho_2'}(T_R;k)) \supset r_R^{-1}(\mathrm{Aug}_m^{\rho_R'}(T_R;k)) \supset \mathrm{Aug}_m^{\rho'}(T;k)\end{aligned}$$

Again, we have $\rho'\leq^G\rho$.

If T is a left cusp q connecting strands $k,k+1$ of T_L. Then clearly $r_L:\mathrm{Aug}_m^{a}(T;k)\xrightarrow{\sim}\mathrm{Aug}_m^{a}(T_L;k)$ is an isomorphism preserving the Ruling stratifications. Hence, by Corollary 4.6, $\rho_L'\leq\rho_L$ implies $\rho_L'\leq^G\rho_L$, which is identical to $\rho'\leq^G\rho$.

If T is n parallel strands with a single base point q on the strand k. Use the notations in Case 3 in the proof of Lemma 3.12, then $\mathrm{Aug}_m^{a}(T;k)\cong\mathrm{Aug}_m^{a}(T_L;k)\times k^*$ via the map $C = (\{(C_l,d_l)\}_{l=0}^1, \{x_l\}_{l=0}^1, H=\{c_r\}) \to ((C_0,d_0),r)$. Also, $\mathrm{Aug}_m^{\tilde{\rho}}(T;k)\cong\mathrm{Aug}_m^{\tilde{\rho}_L}(T_L;k)\times k^*$ for all $\tilde{\rho}\in\mathrm{NR}_T^m$, via the same map. Thus, under this identification, $\rho_L'\leq\rho_L$ implies that

$$\begin{aligned}\overline{\mathrm{Aug}_m^{\rho}(T;k)} &= \overline{\mathrm{Aug}_m^{\rho_L}(T_L;k)}\times k^*\\ \supset\ & \mathrm{Aug}_m^{\rho_L'}(T_L;k)\times k^* = \mathrm{Aug}_m^{\rho'}(T;k)\end{aligned}$$

That is, $\rho'\leq^G\rho$.

If T is a right cusp q connecting strands $k,k+1$ of T_R. Use the notations in Case 4 in the proof of Lemma 3.12, then

$$\mathrm{Aug}_m^{a}(T;k)\cong\{(C_0,d_0)\in\mathrm{Aug}_m^{a}(T_L;k) | < d_0e_k, e_{k+1} >= 0\}\times k^{\beta}$$

via the map $C = (\{(C_l,d_l)\}_{l=0}^2, \{x_l\}_{l=0}^2, H=\{H_r\}) \to ((C_0,d_0),r)$, where $\beta=0$ (resp. 1) if $m\neq 0$ (resp. $m=0$). Notice that for any $\tilde{\rho}\in\mathrm{NR}_T^m$, $\tilde{\rho_L}$ pairs the strands $k,k+1$, i.e. $(kk+1)\in\tilde{\rho}_L$. It follows that any $(C_0,d_0)\in\mathrm{Aug}_m^{\tilde{\rho}_L}(T_L;k)$ automatically satisfies $< d_0e_k, e_{k+1} >$. Therefore, via the same map as above, we have the identification

$$\mathrm{Aug}_m^{\tilde{\rho}}(T;k)\cong\mathrm{Aug}_m^{\tilde{\rho}_L}(T_L;k)\times k^{\beta}$$

for all $\tilde{\rho}\in\mathrm{NR}_T^m$. Now, under this identification, $\rho_L'\leq\rho_L$ implies that

$$\overline{\mathrm{Aug}_m^{\rho}(T;k)} = \overline{\mathrm{Aug}_m^{\rho_L}(T_L;k)}\times k^{\beta}$$

$$\supset \quad \mathrm{Aug}_m^{\rho'_L}(T_L;k)\times k^{\beta} = \mathrm{Aug}_m^{\rho'}(T;k)$$

That is, $\rho' \leq^G \rho$.

□

4.3. **A conjecture for the general case.** Now, we consider the general case. As in Section 1.4.4, suppose $T = E_1 \circ E_2 \circ \ldots \circ E_n$ is a composition of n elementary Legendrian tangles. For simplicity, *denote* $L_i := (E_{i+1})_L$ for $0 \leq i \leq n-1$ and $L_n := (E_n)_R$. Let $r_i : \mathrm{Aug}_m^a(T;k) \to \mathrm{Aug}_m^a(L_i;k)$ be the obvious restriction map, for $0 \leq i \leq n$. For any m-graded normal ruling ρ of T, *denote* $r_i(\rho) := \rho|_{L_i}$ for $0 \leq i \leq n$.

Similar to Definition 2.1, we define an *algebraic partial order* on the set NR_T^m of m-graded normal rulings of T:

Definition 4.16. Given any $\rho, \rho' \in \mathrm{NR}_T^m$, we say $\rho' \leq^A \rho$ (or $\rho' \leq \rho$) if $r_i(\rho') \leq^A r_i(\rho)$ (Definition 4.4) for $0 \leq i \leq n$.

Notice that by Corollary 4.6, this definition is compatible with Definition 2.1, in the sense that $(\mathrm{NR}_T^m, \leq^A)$ restricts to $(\mathrm{NR}_T^m(\rho_L, \rho_R), \leq^A)$ in Definition 2.1.

Conjecture 4.17. *The full ruling decomposition* $\mathrm{Aug}_m^a(T;k) = \sqcup_{\rho\in\mathrm{NR}_T^m}\mathrm{Aug}_m^{\rho}(T;k)$ *is a Whitney stratification. Moreover, given any two m-graded normal rulings ρ', ρ for T, $\rho' \leq^G \rho$ if and only if the $\rho' \leq^A \rho$, that is:*

$$\overline{\mathrm{Aug}_m^{\rho}(T;k)} = \sqcup_{\rho'\in\mathrm{NR}_T^m:\rho'\leq^A\rho}\mathrm{Aug}_m^{\rho'}(T;k) \tag{4.3.1}$$

where the closure is taken in $\mathrm{Aug}_m^a(T;k)$.

This would give an combinatorial description of the geometric partial order on the ruling decomposition in the general case.

Remark 4.18. By the previous two subsections, the conjecture holds for the 'building blocks' of Legendrian tangles: the trivial and elementary Legendrian tangles.

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